# Overview of Stabilizers and Magic

**MBQM 18/11/2024**

Mark Howard, University of Galway, Ireland

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$$D(a,b)D(a',b') = D(a',b')D(a,b) \leftrightarrow \begin{cases} a'b^T + b'a^T = 0 \bmod 2 \\ \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} 0 & \mathbb{I}_n \\ \mathbb{I}_n & 0 \end{bmatrix} \begin{bmatrix} (a')^T \\ (b')^T \end{bmatrix} \end{cases}$$

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$$\text{Cliff}_{2^n}/WH_{2^n} \approx \text{Sp}(2n, \mathbb{F}_2) = \left\langle \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\rangle$$

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Stabilizer circuits (Cliffords + Pauli Mmts) on stabilizer input are efficiently/$poly(n)$ simulable on a classical computer

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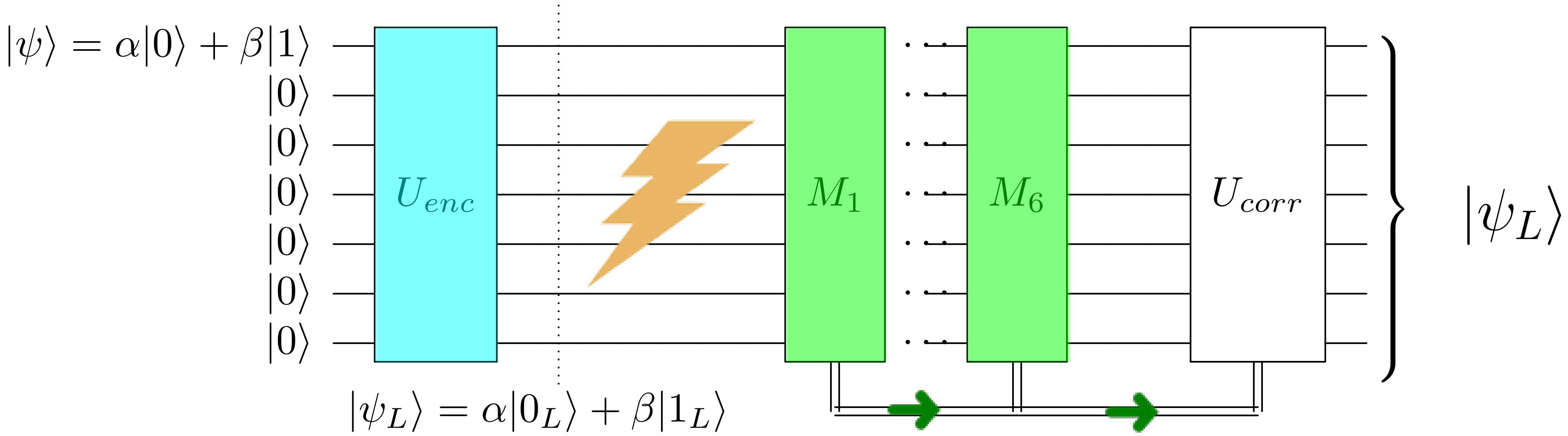


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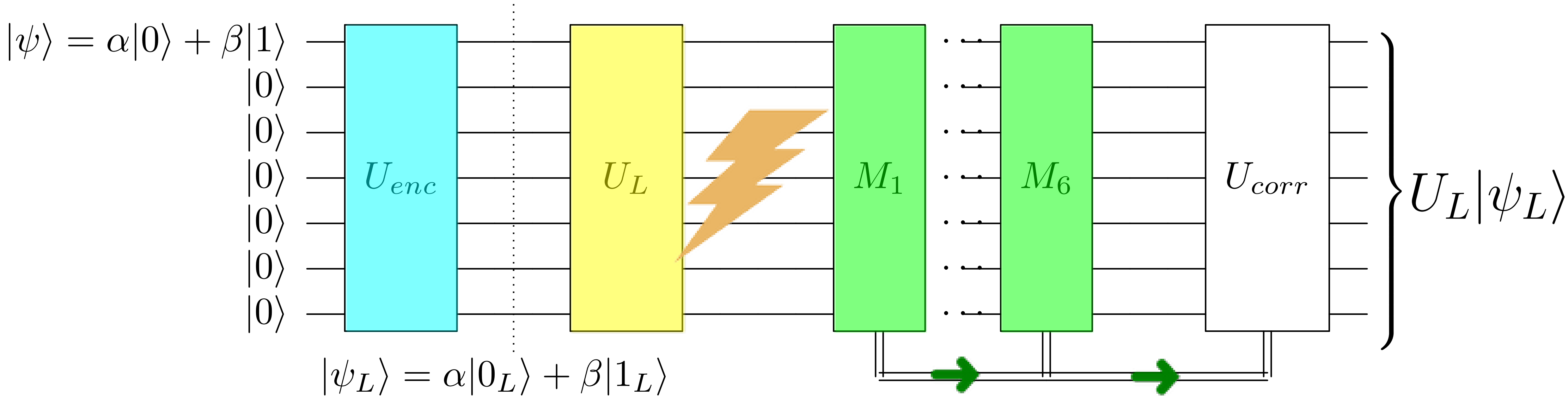


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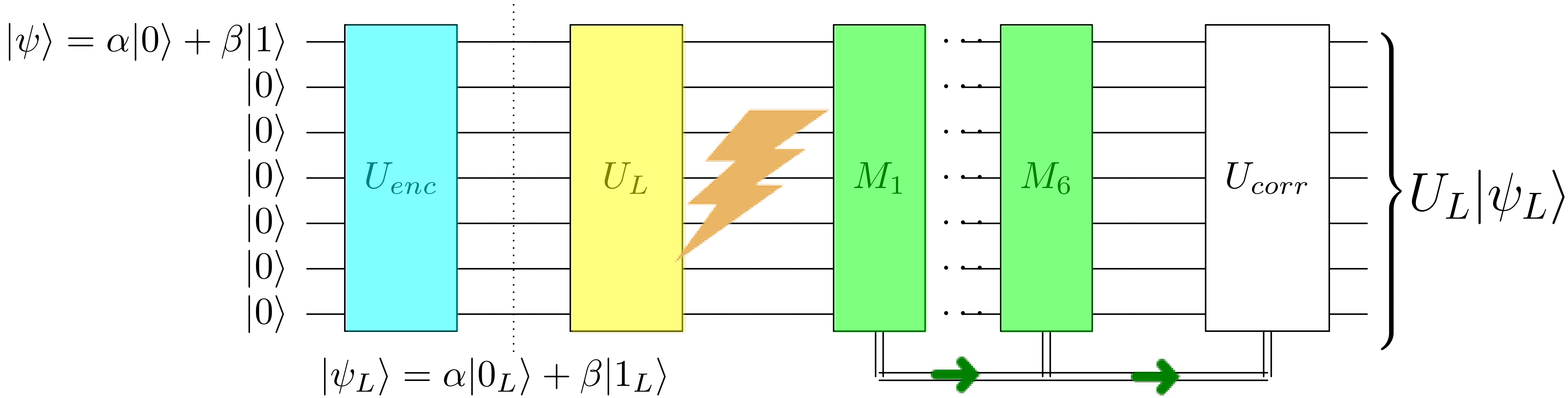


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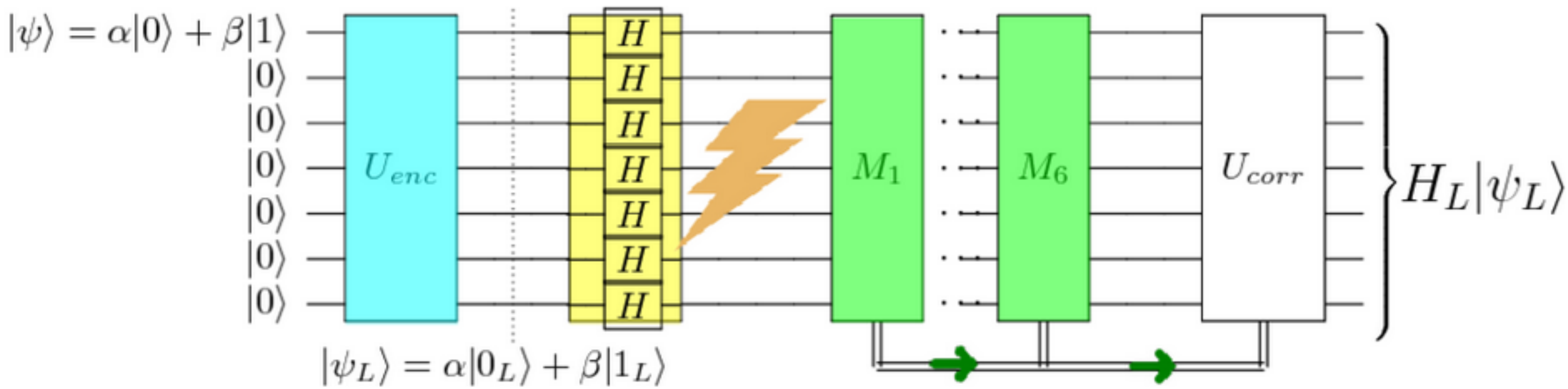


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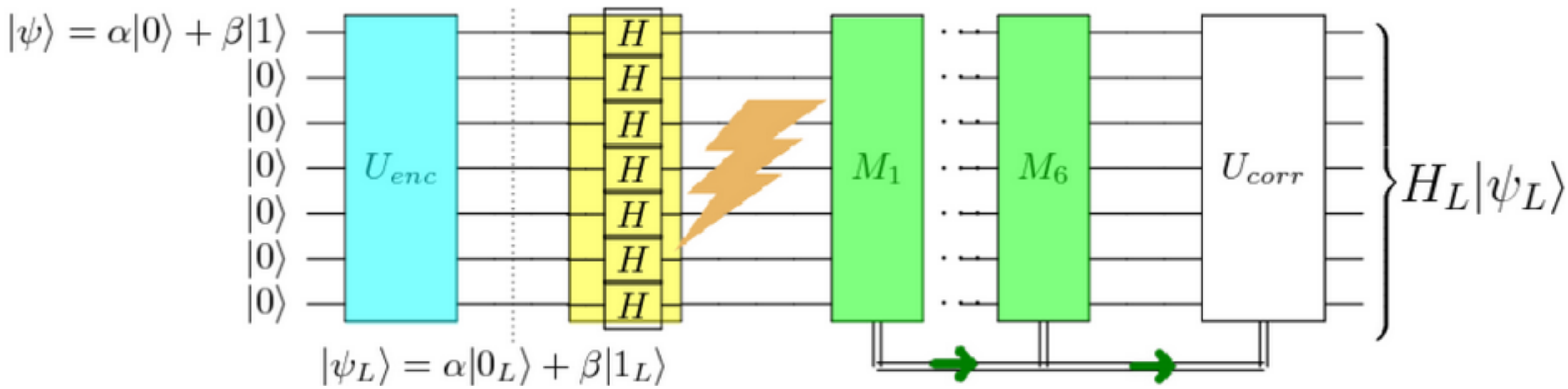


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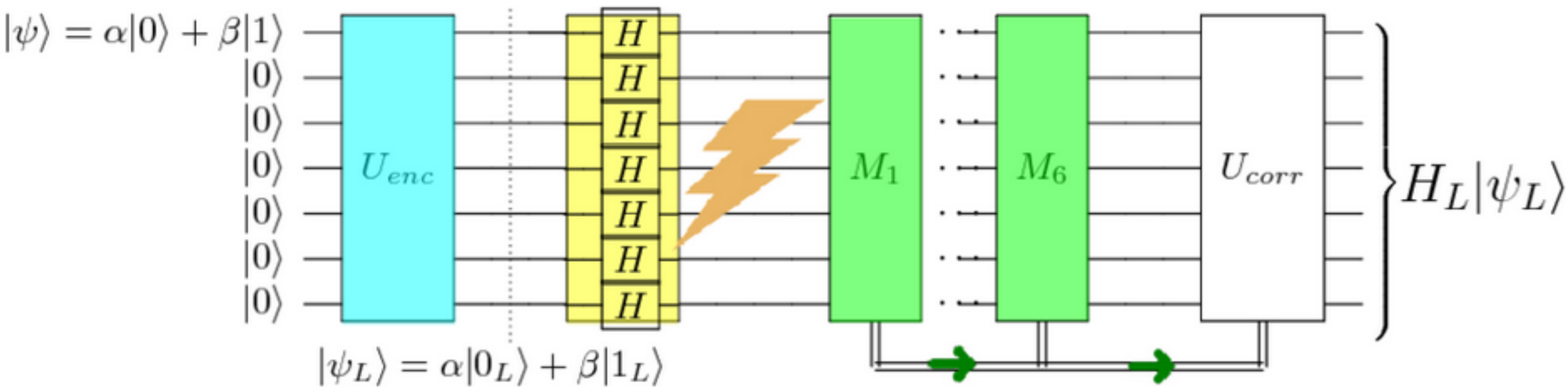


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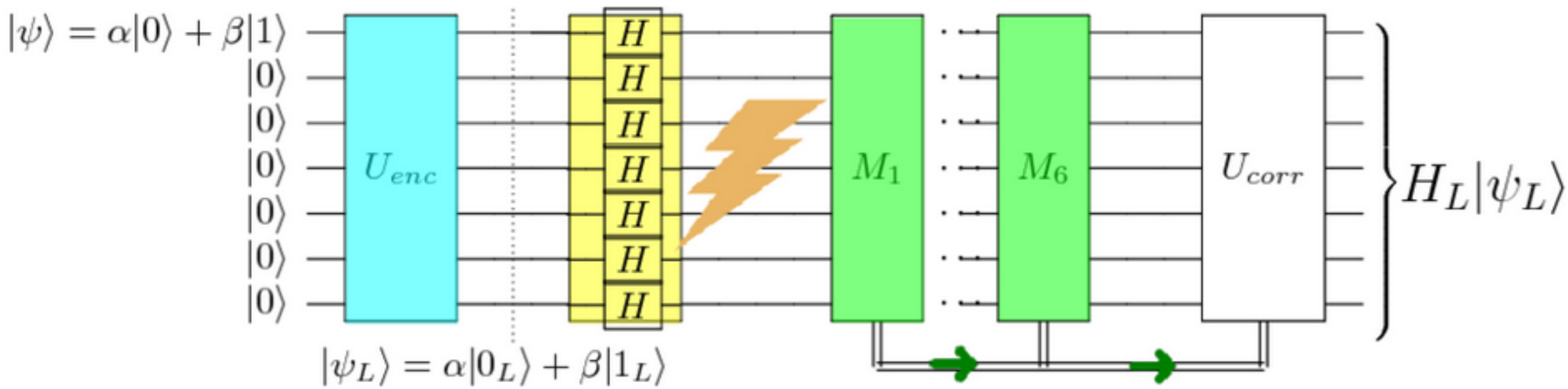


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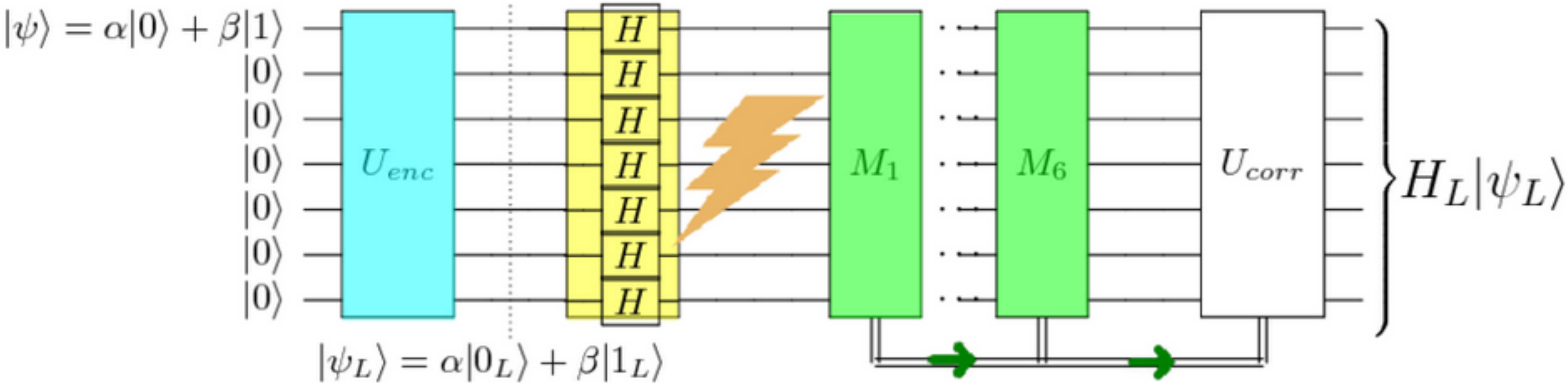


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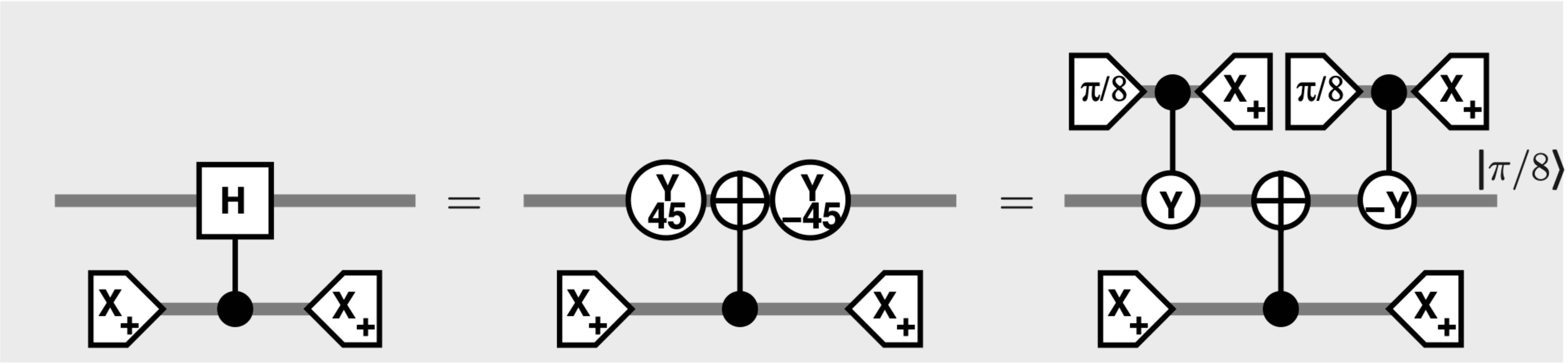


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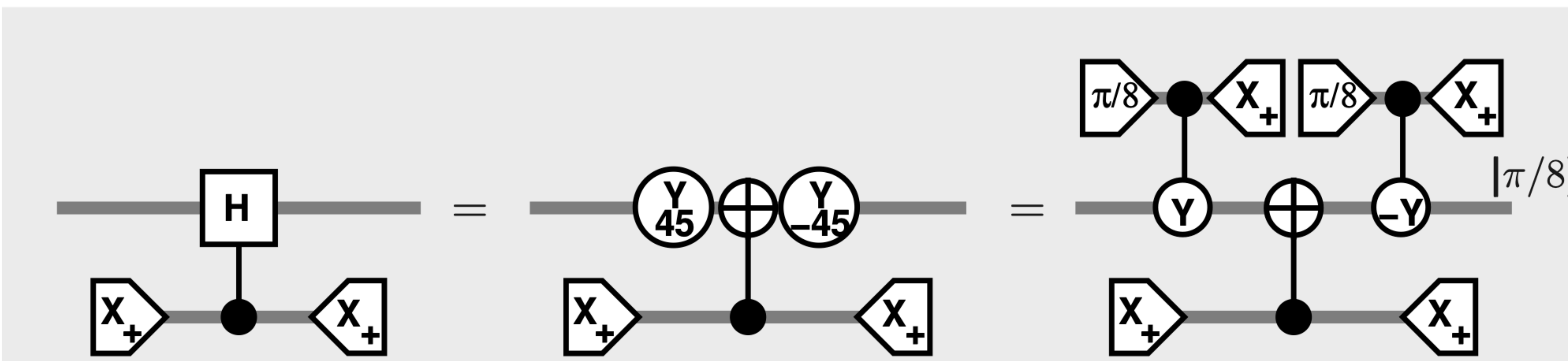


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Consider preparation of logical $|\pi/8\rangle$ states. A version of the purification scheme for $|\pi/8\rangle$ states given in [1] is analyzed by Bravyi and Kitaev [30] in the context of "magic states distillation". They show that magic states, which include $|\pi/8\rangle$, are distillable given a way of preparing them with probability of error below about 35 %, assuming no error in Clifford group operations.

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arXiv:quant-ph/0403025 [pdf, ps, other] quant-ph doi 10.1103/PhysRevA.71.022316

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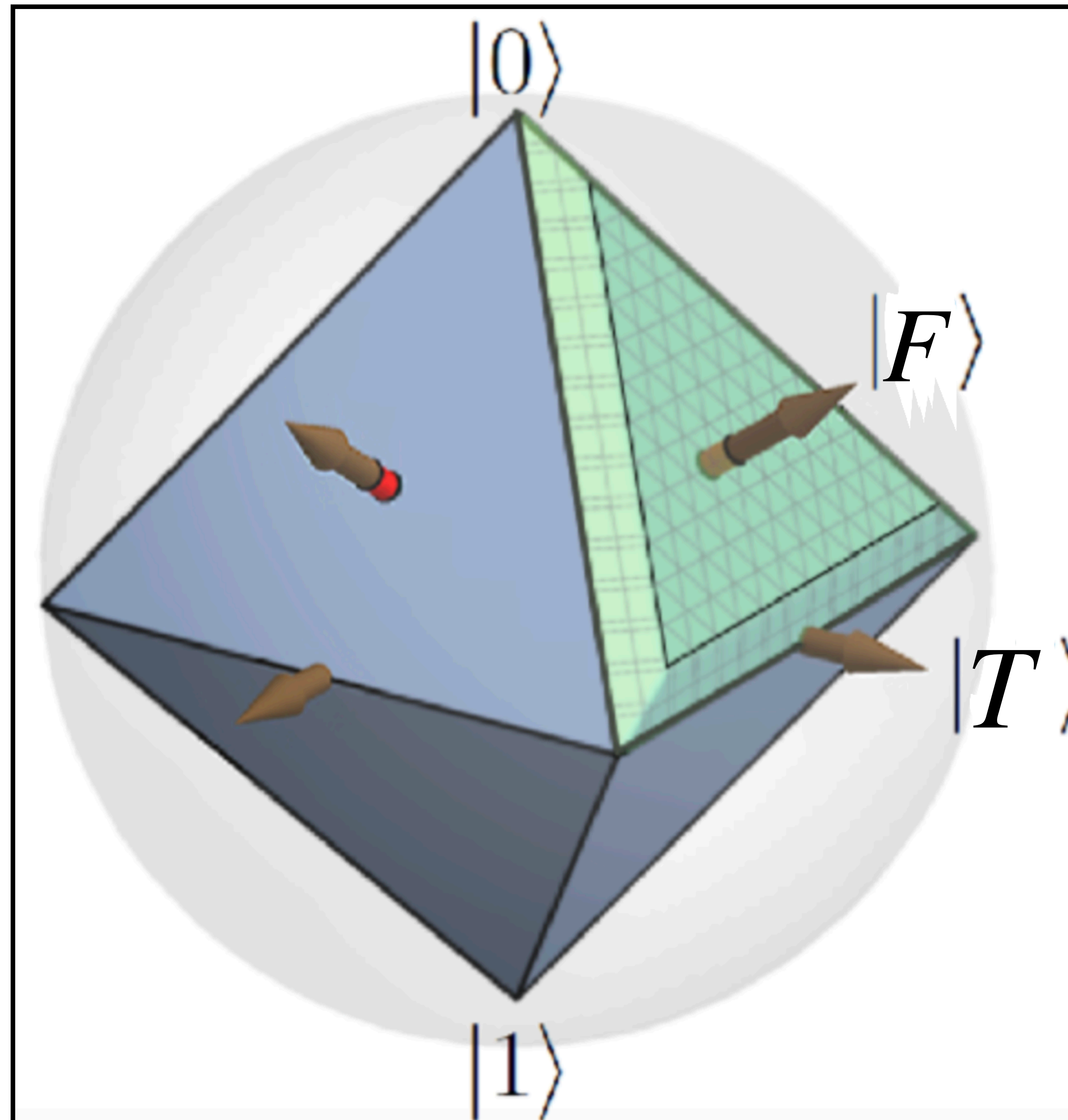

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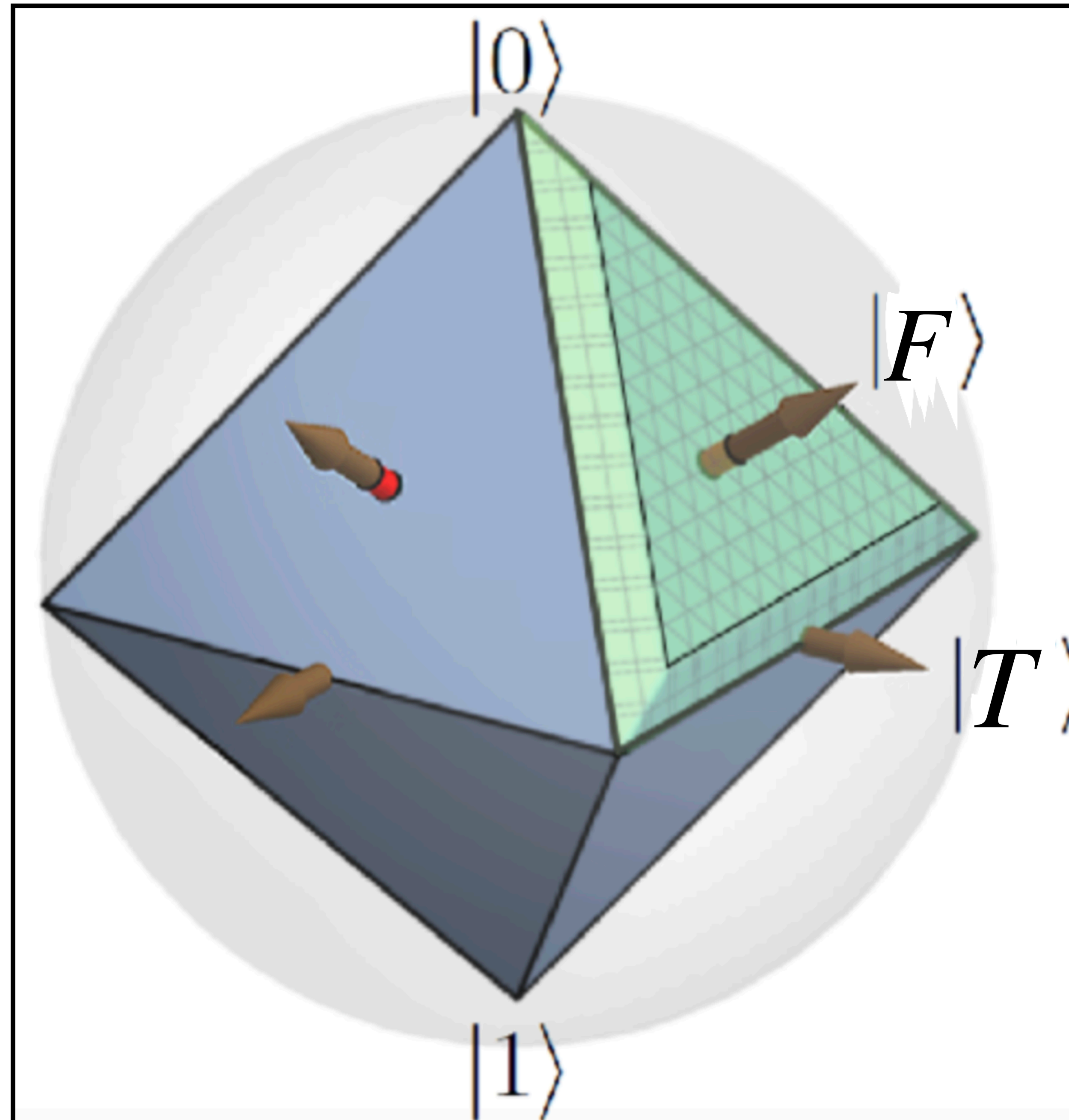

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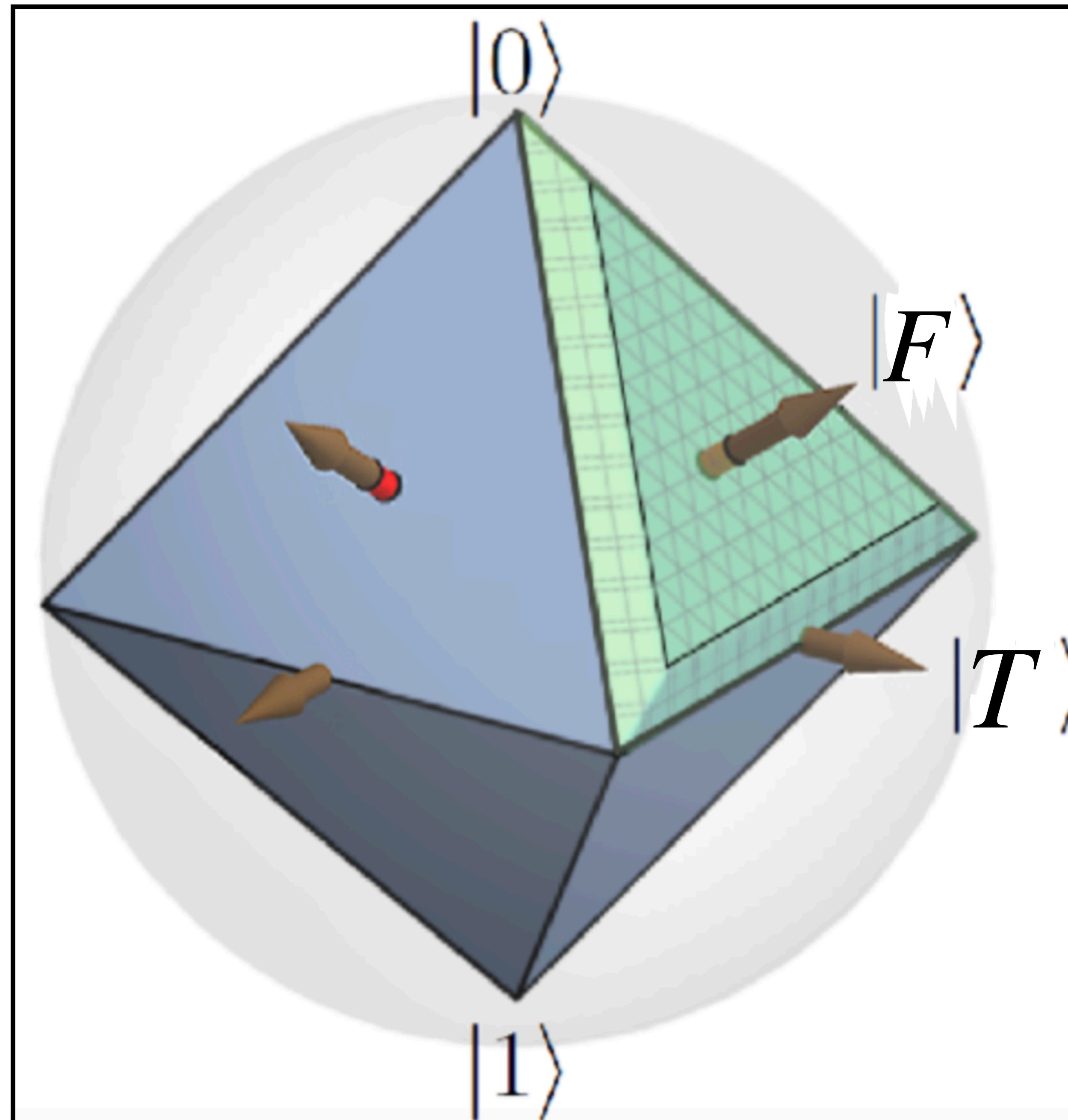

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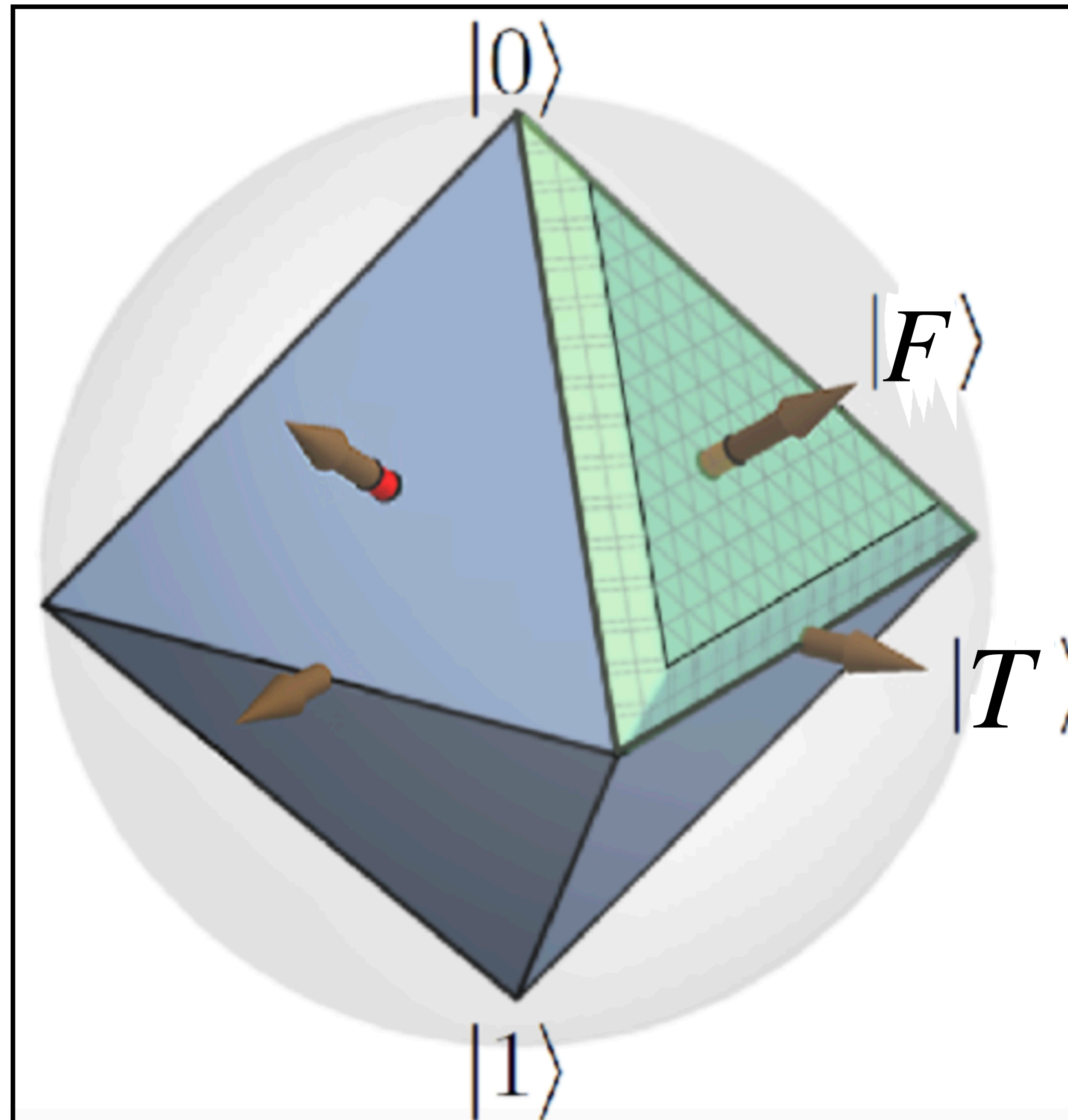
  2. Threshold Error rates



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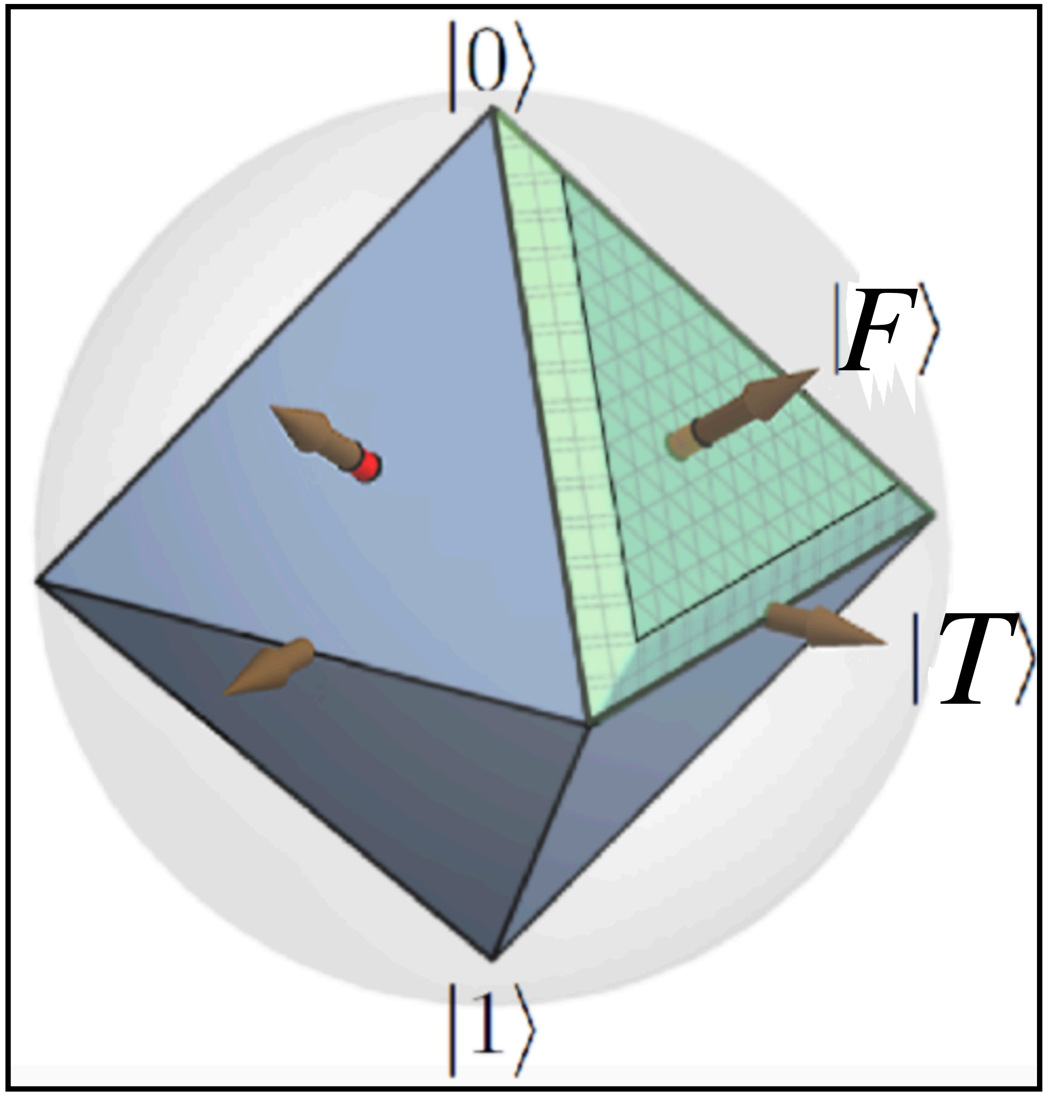

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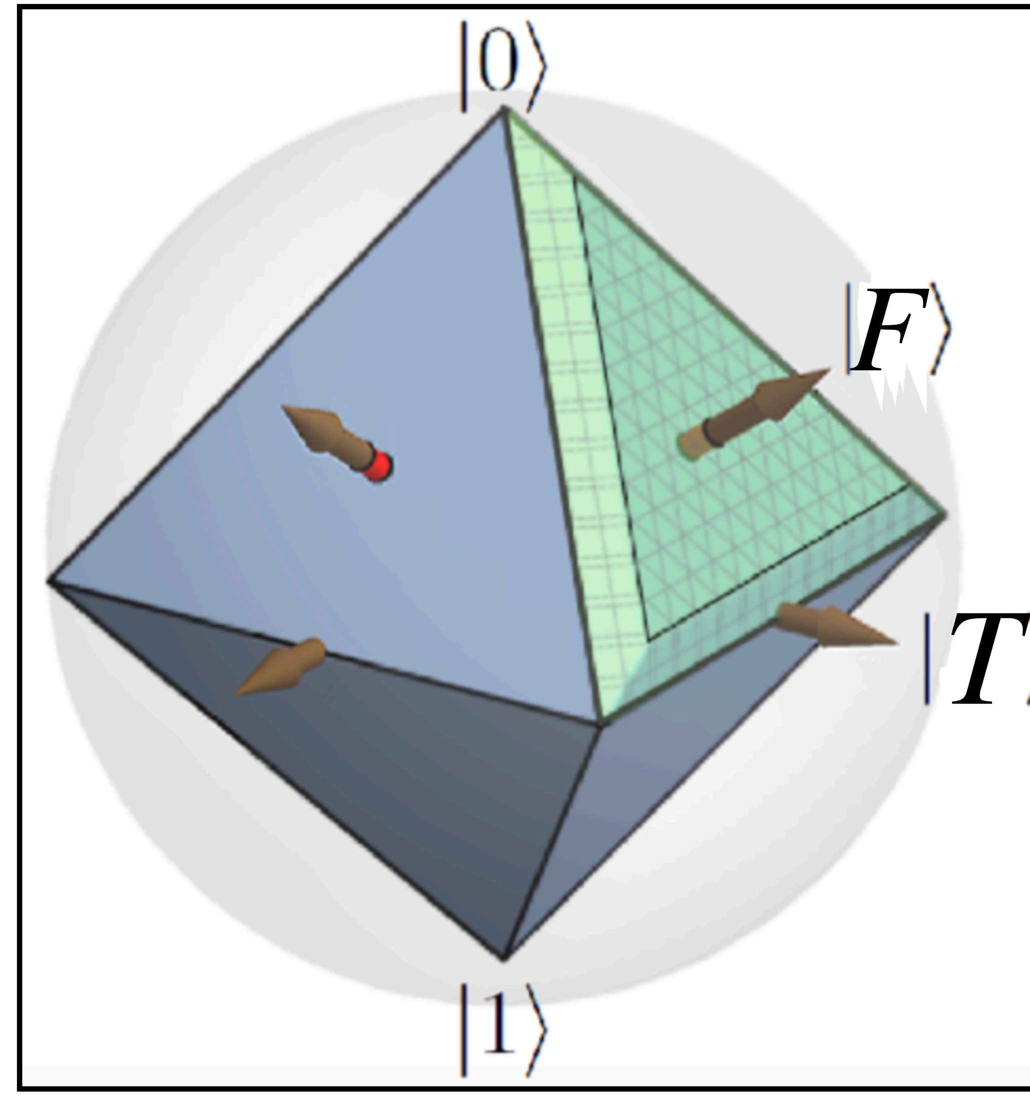

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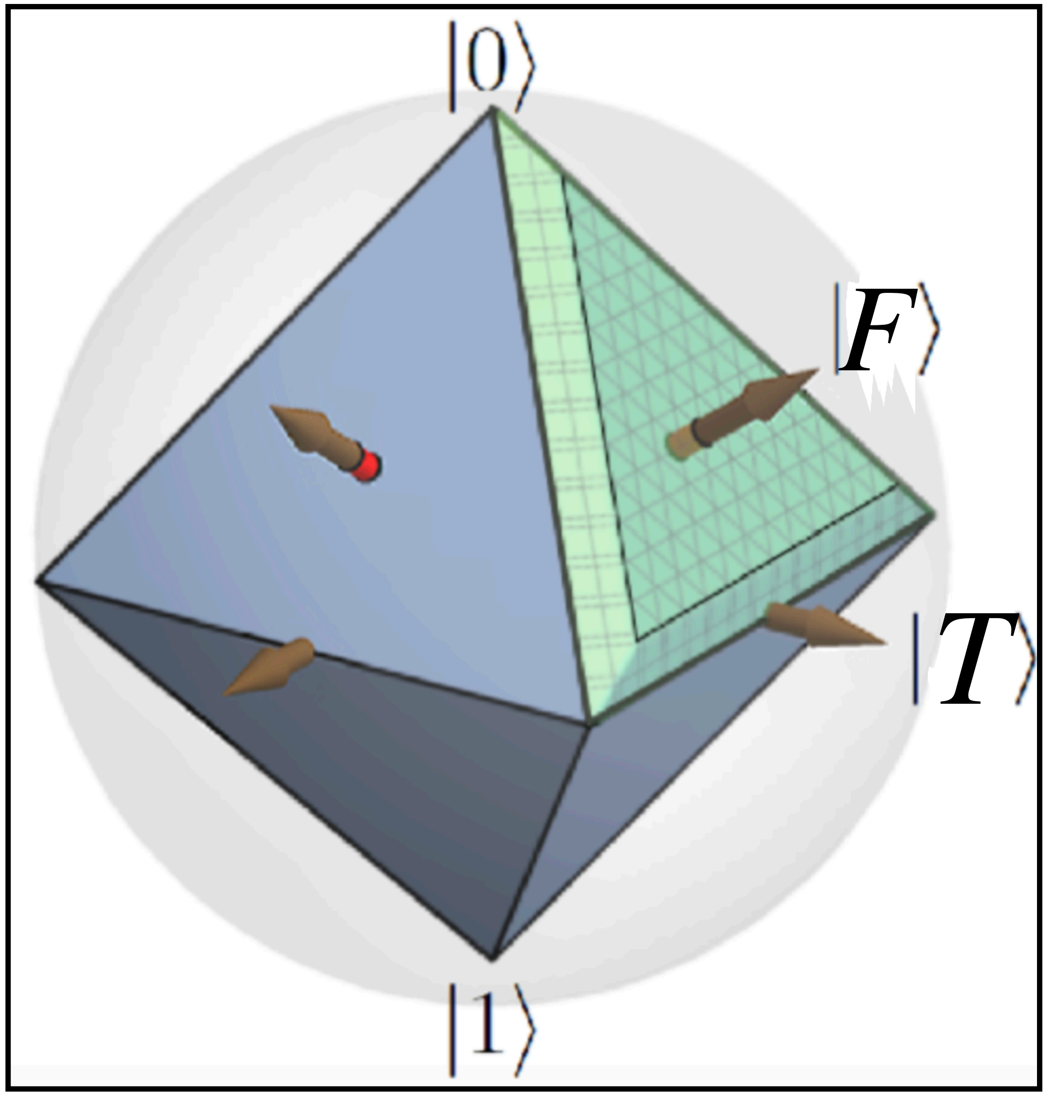


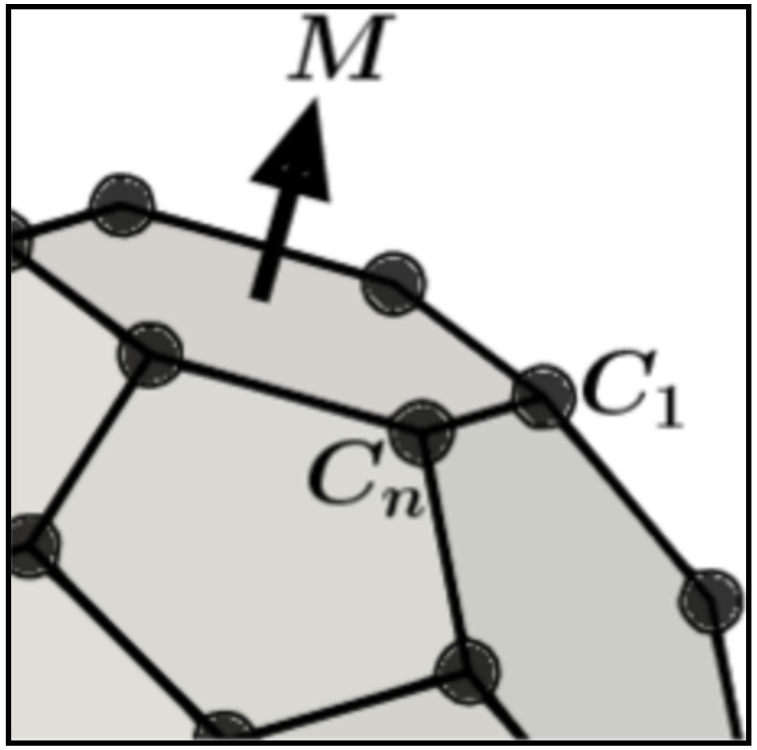

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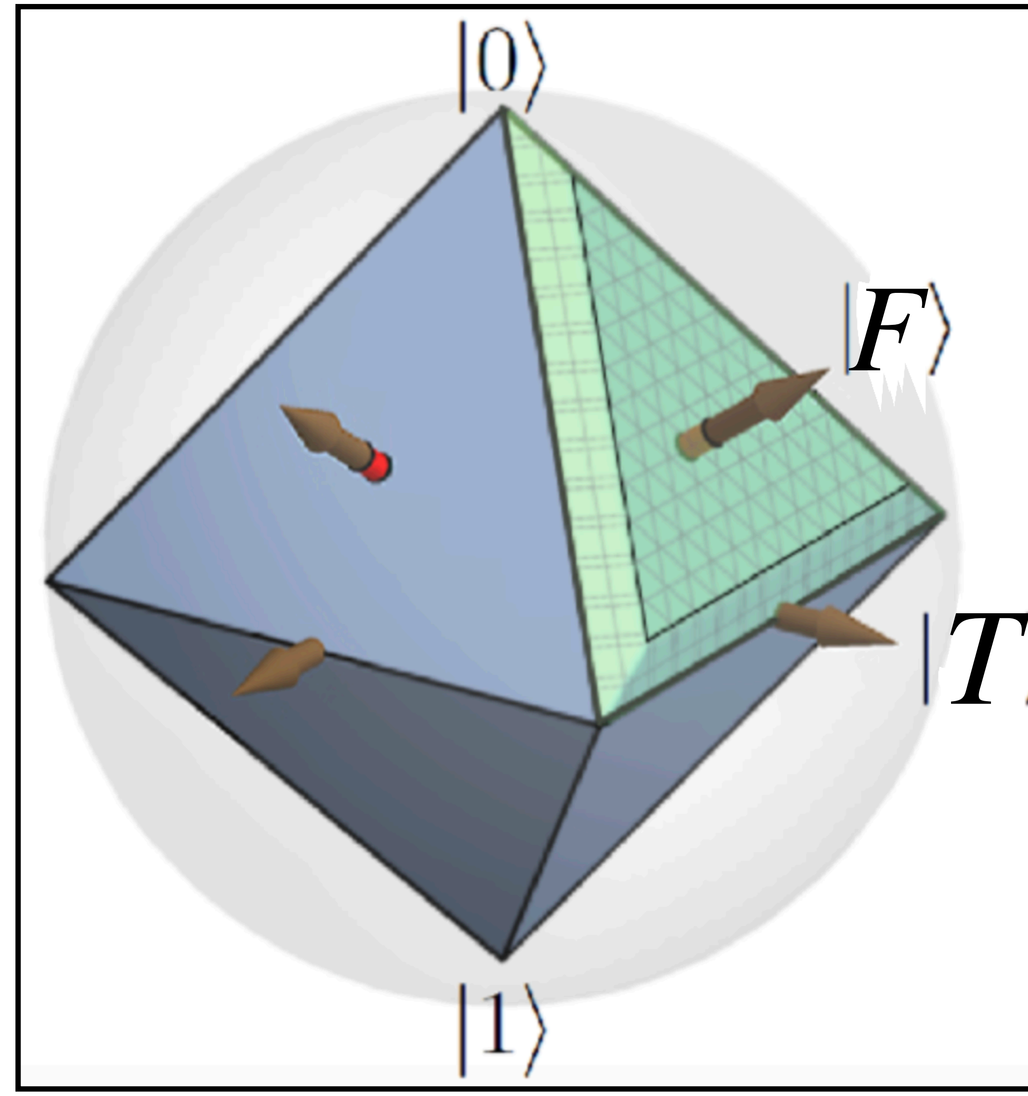


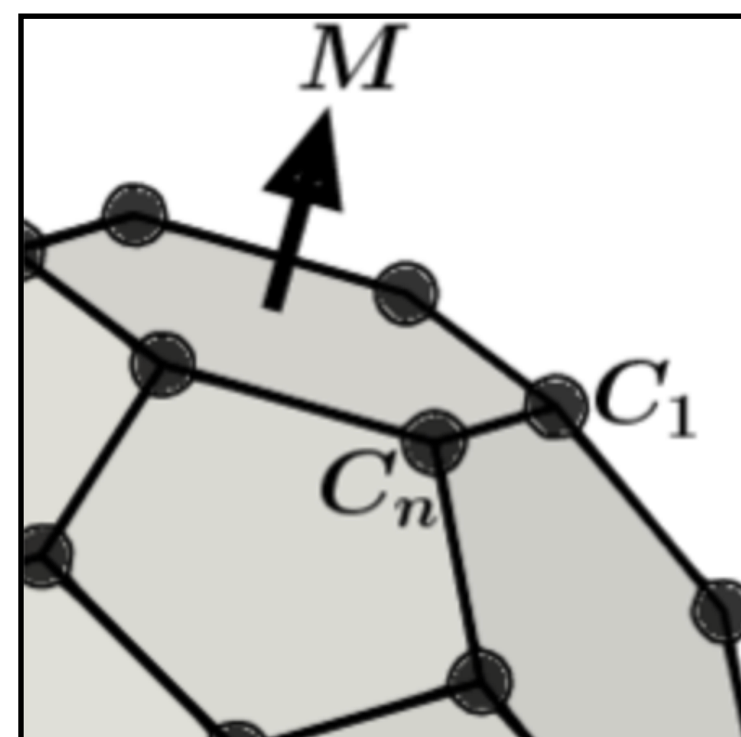

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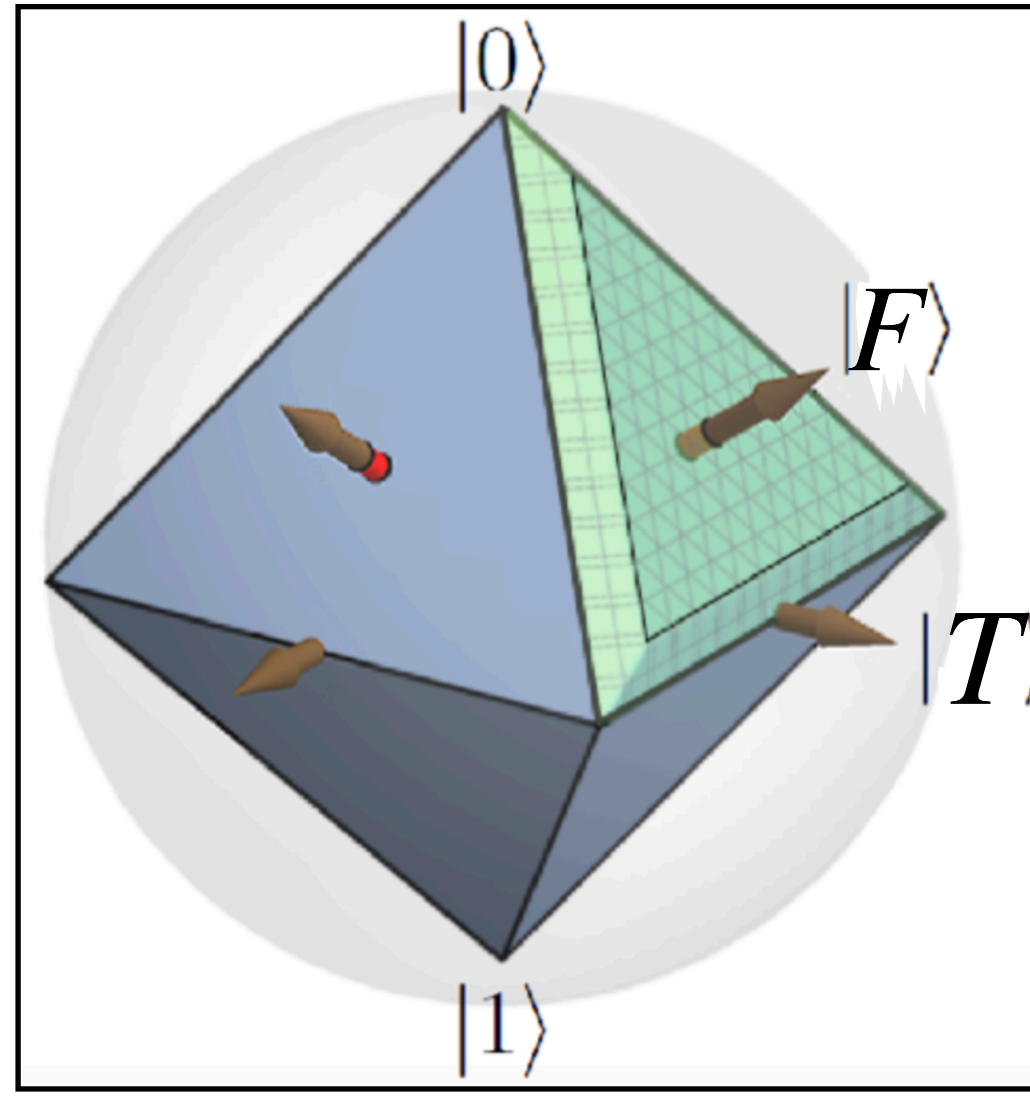


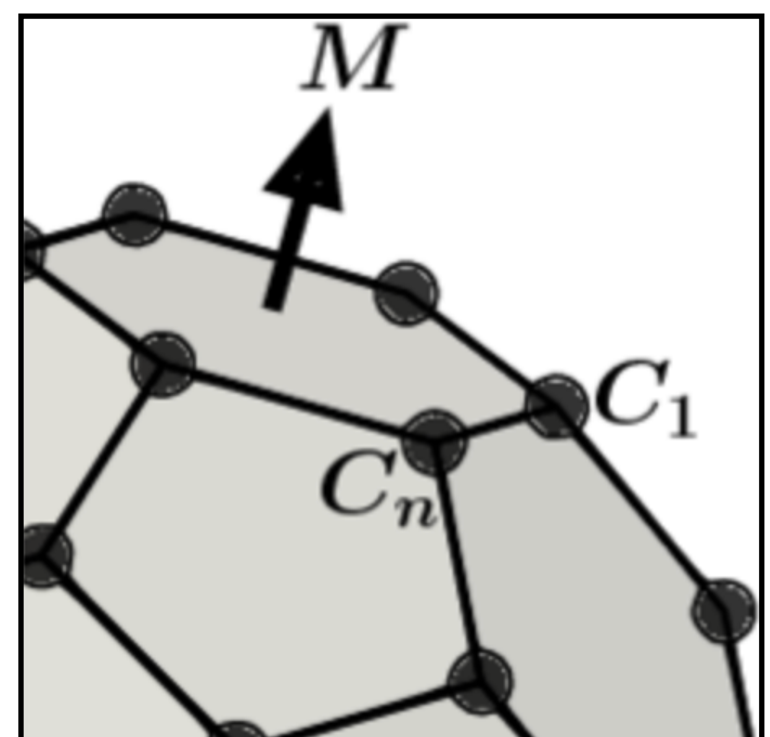

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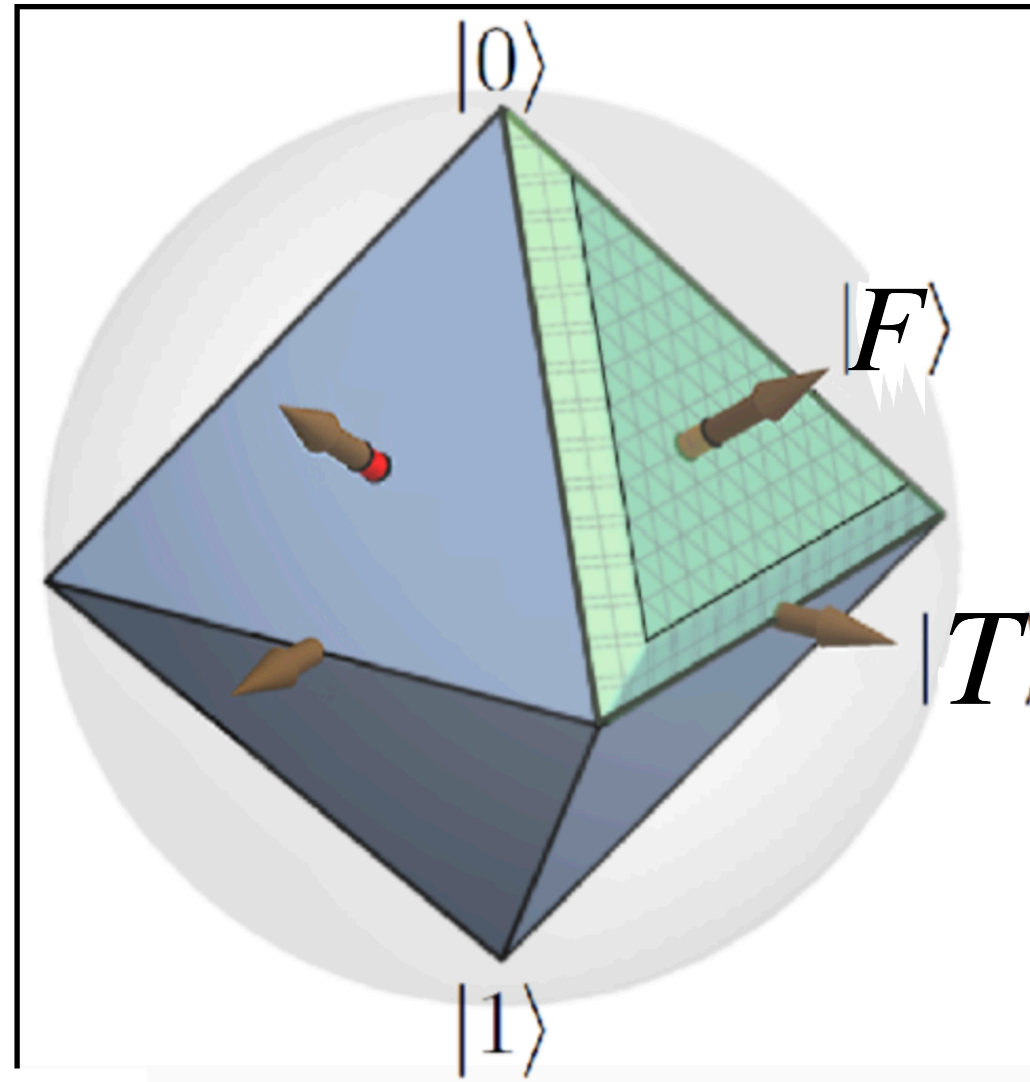


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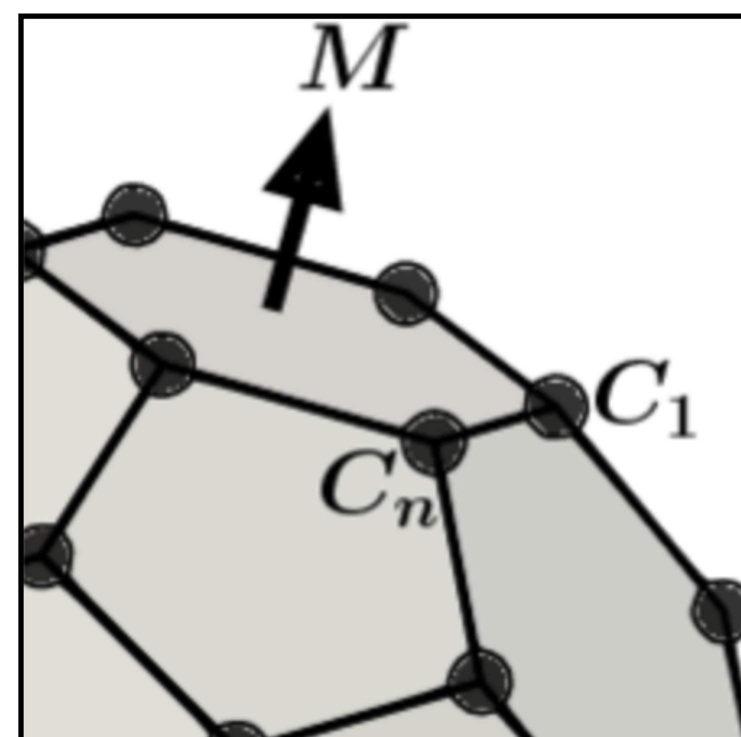

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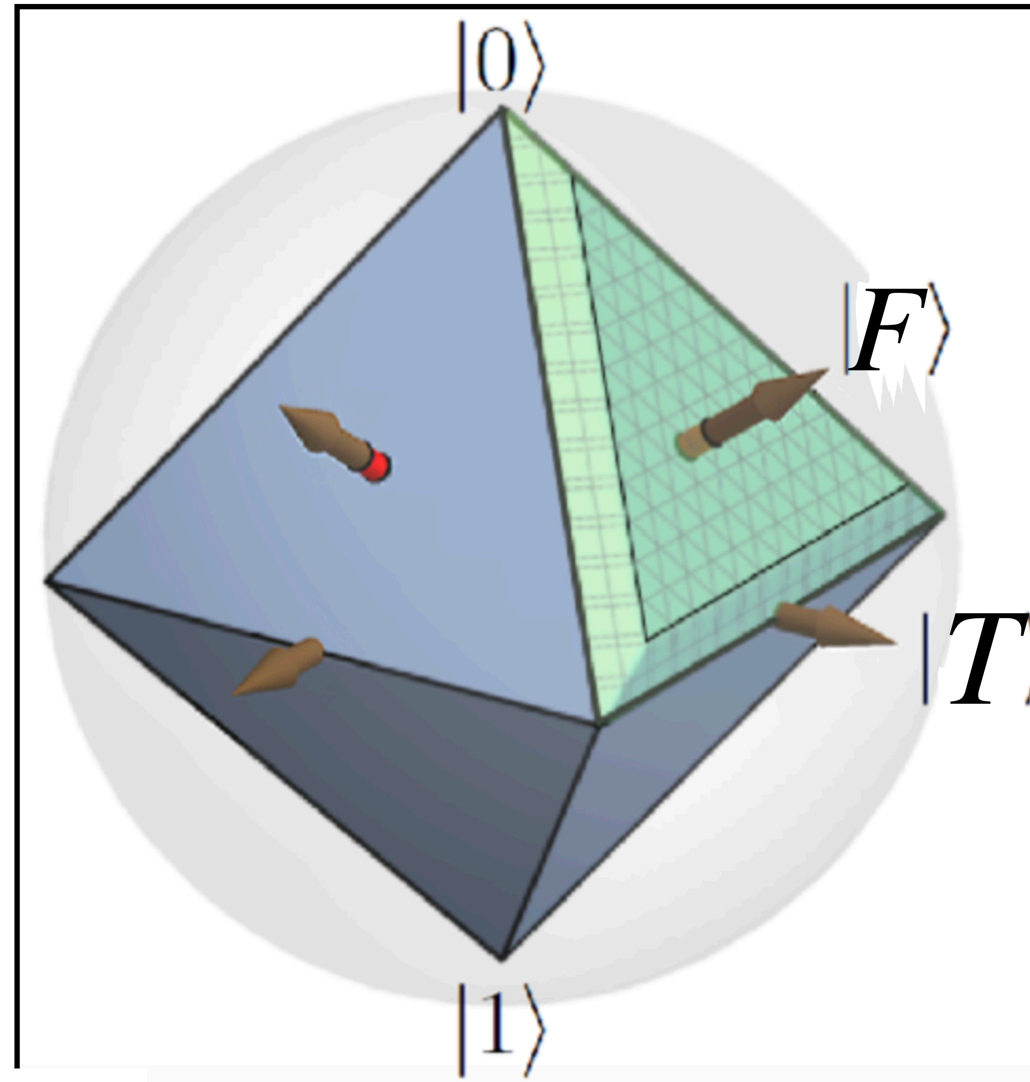


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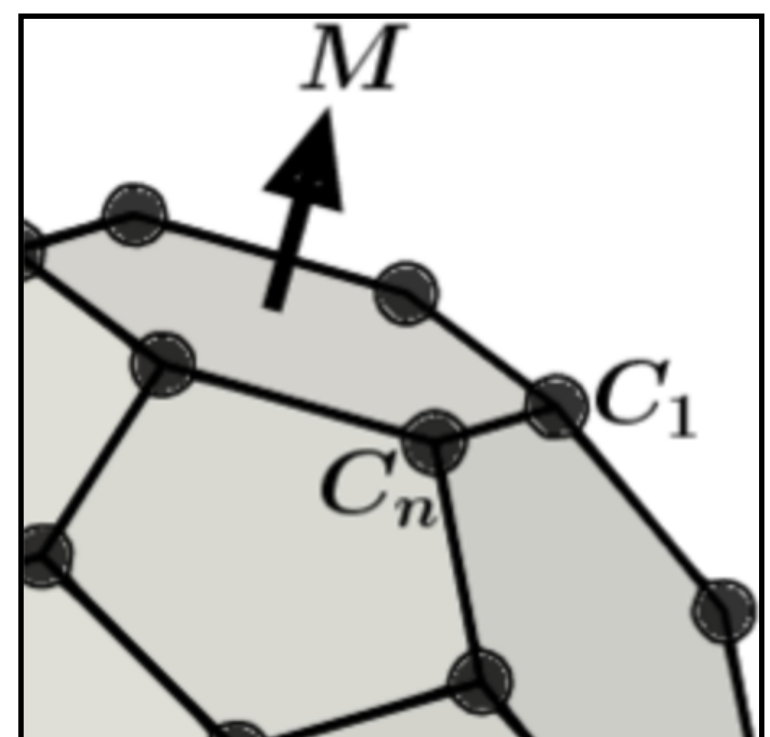

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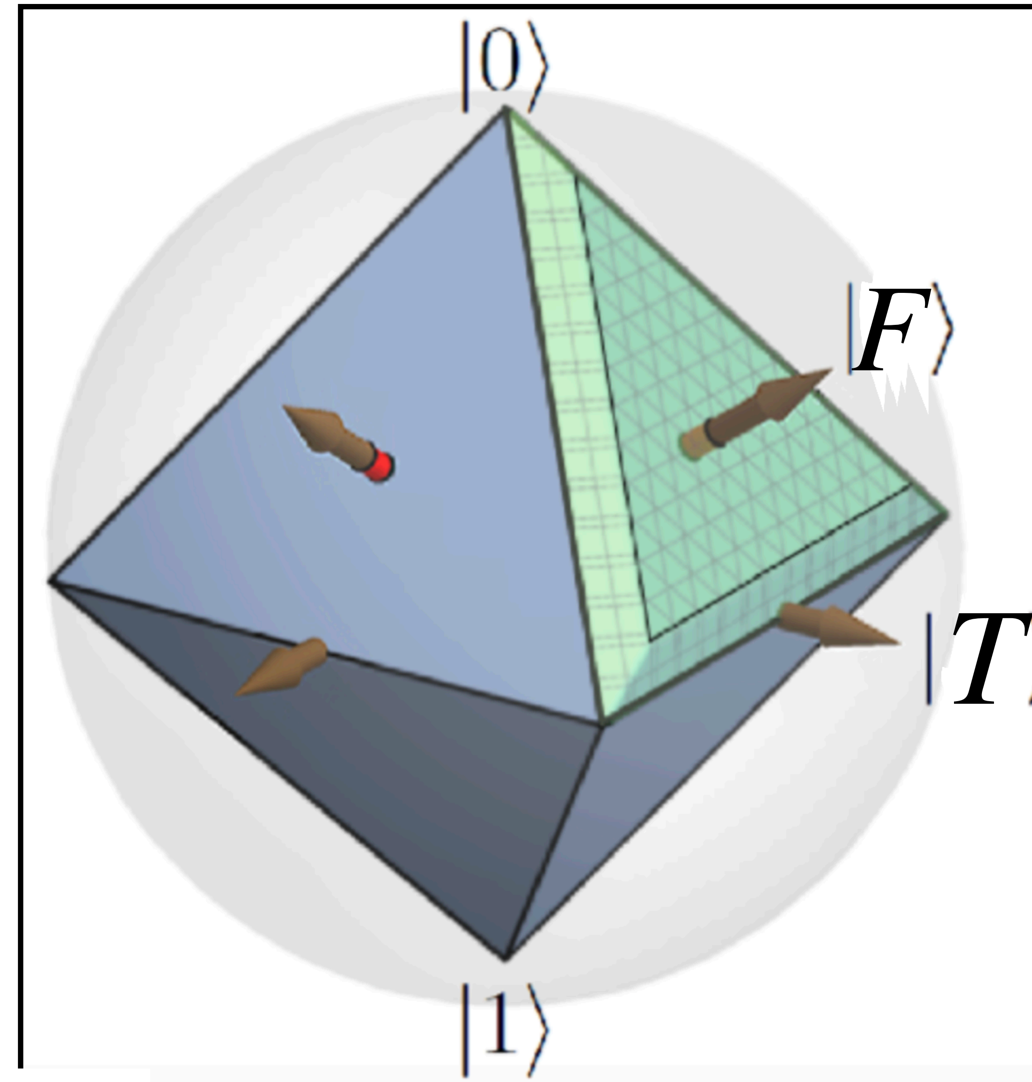


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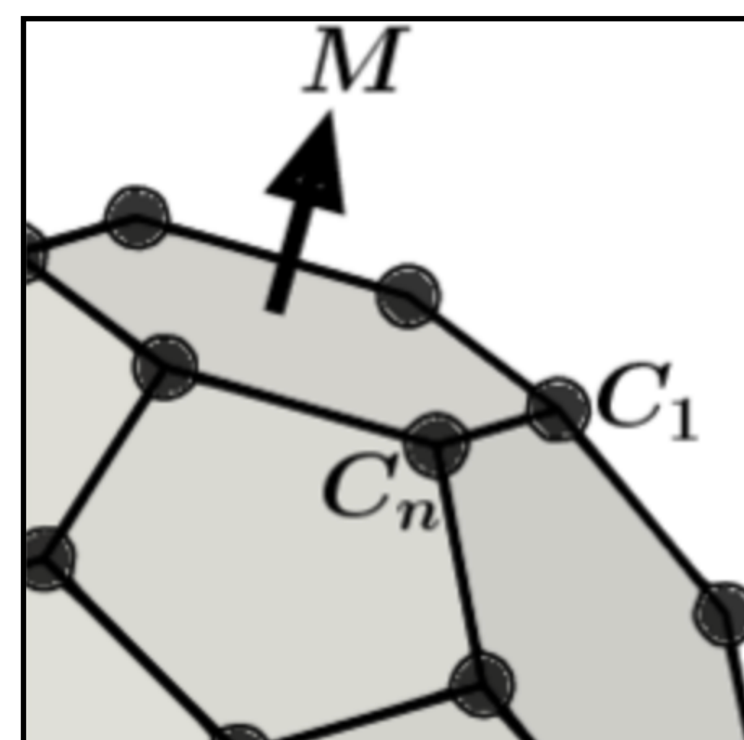


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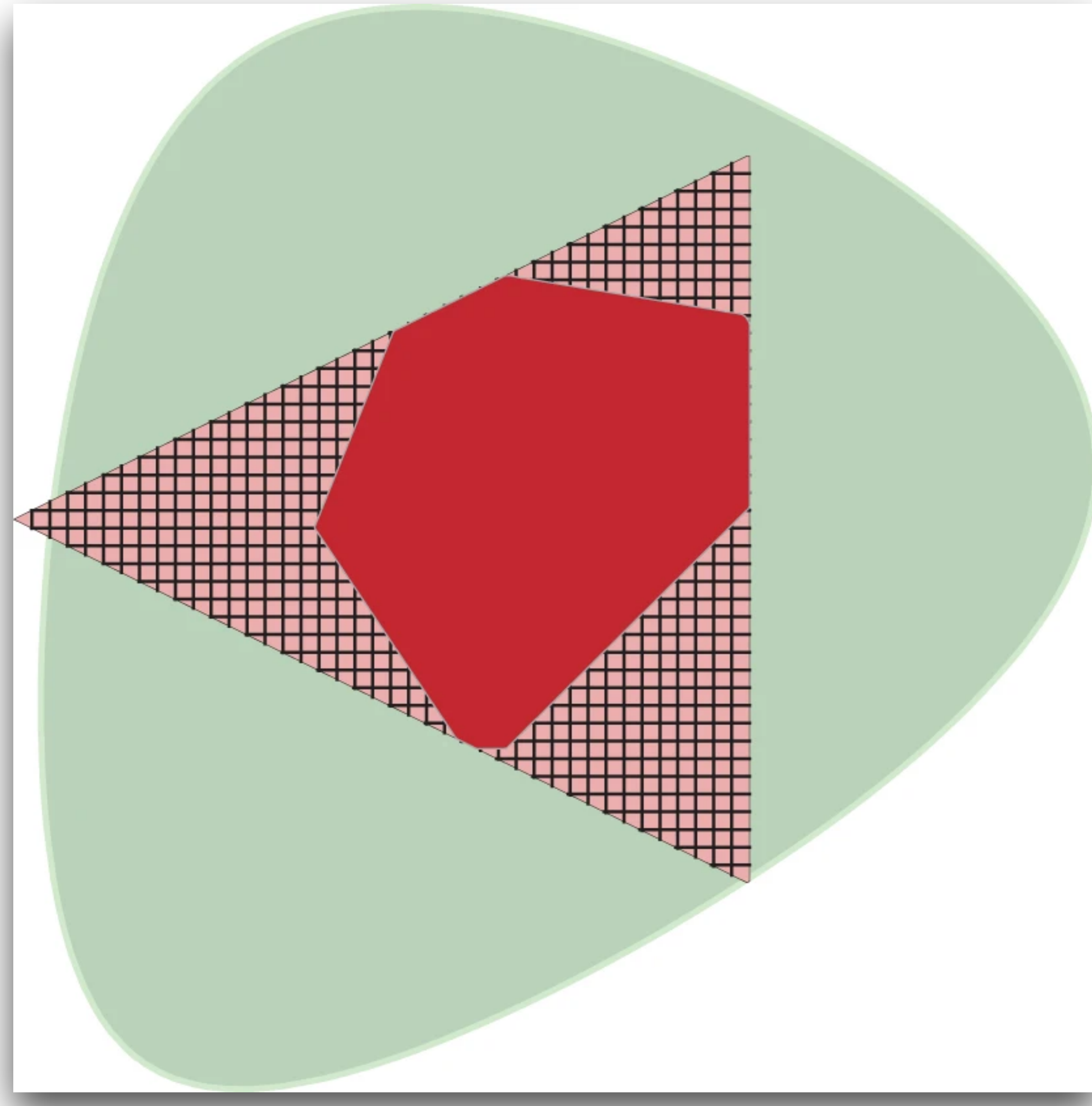

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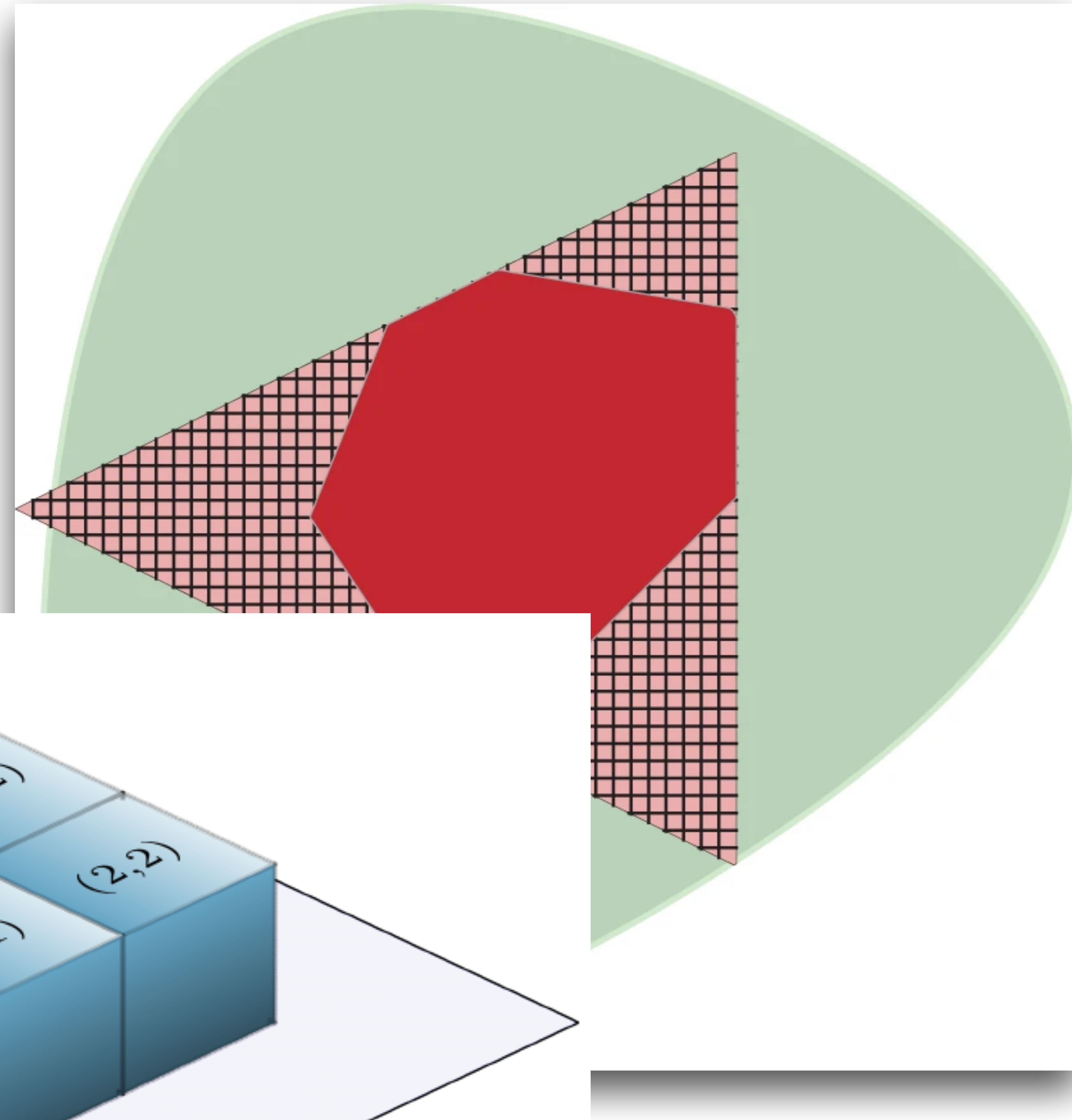

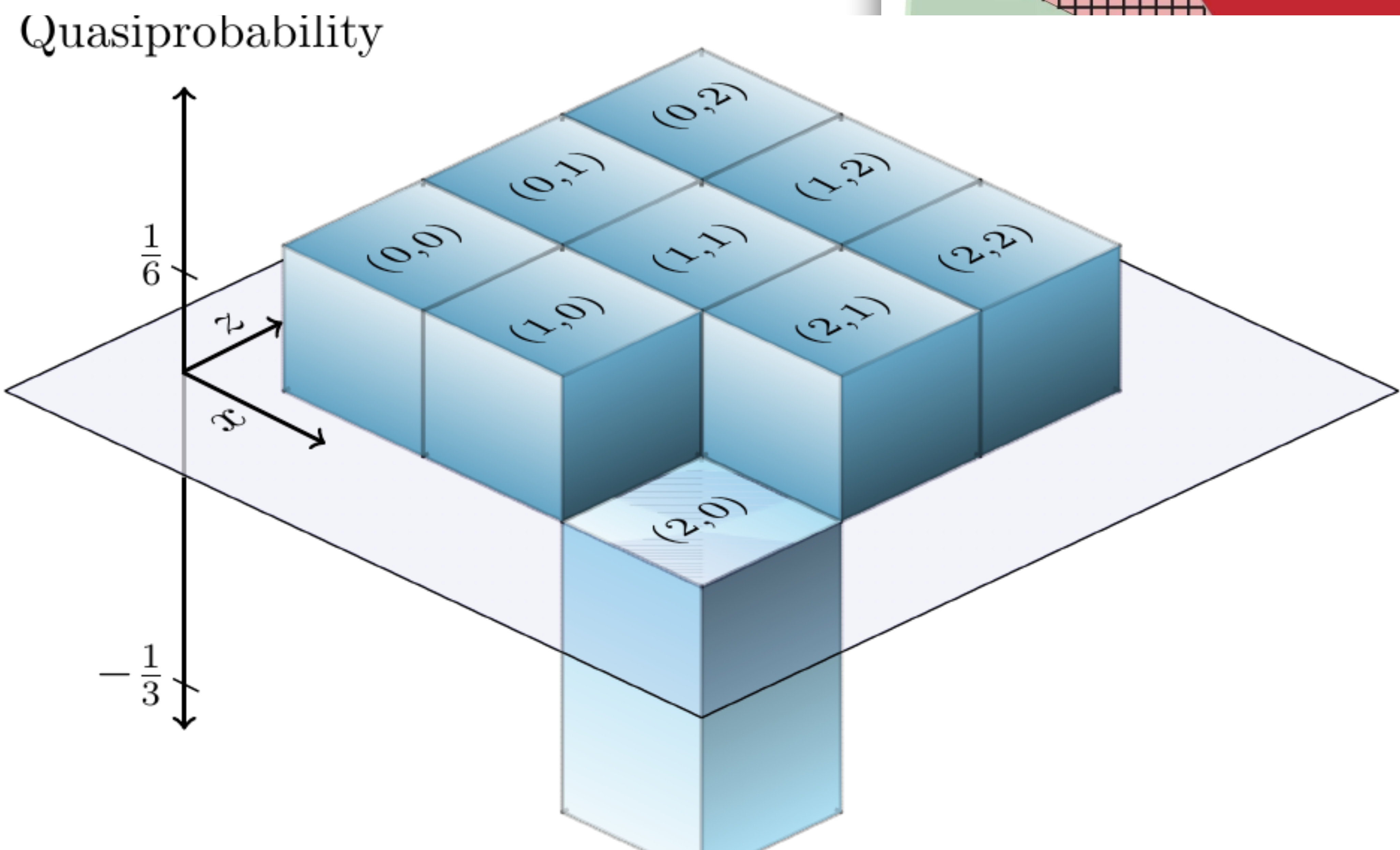

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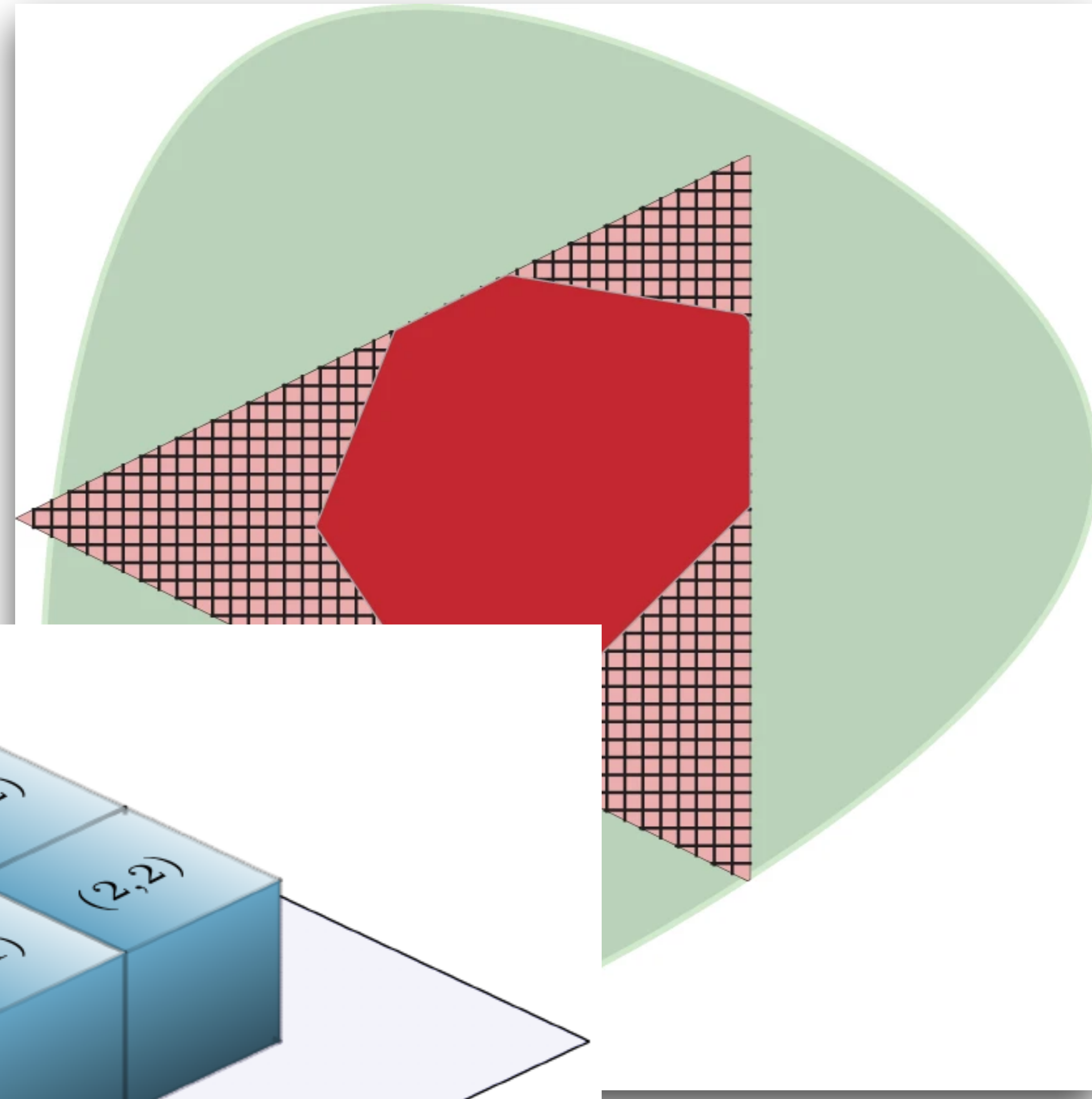

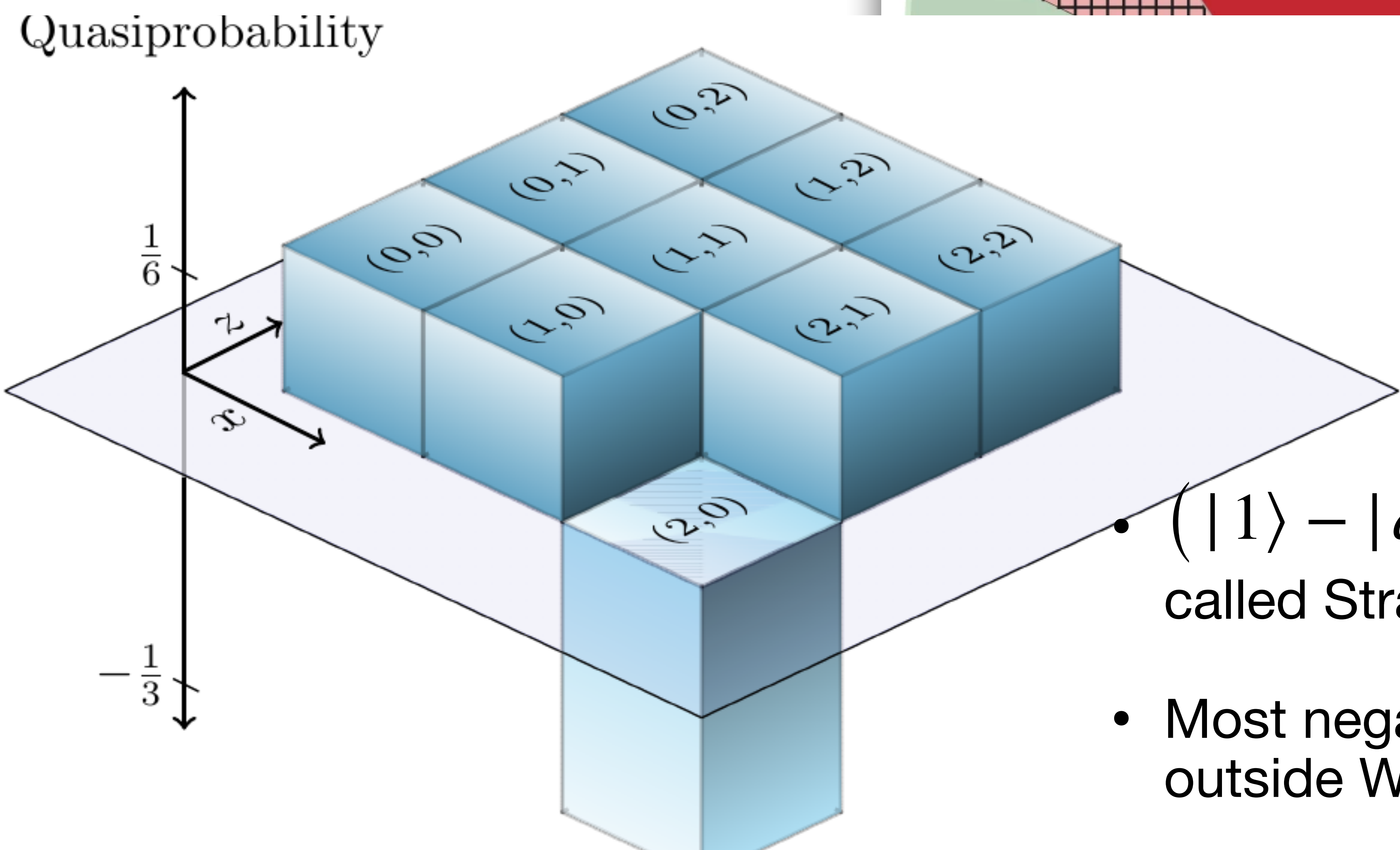


- $\left(|1\rangle - |d-1\rangle\right)/\sqrt{2}$ often called Strange states $|\mathbb{S}\rangle$
- Most negative, furthest outside Wigner polytope

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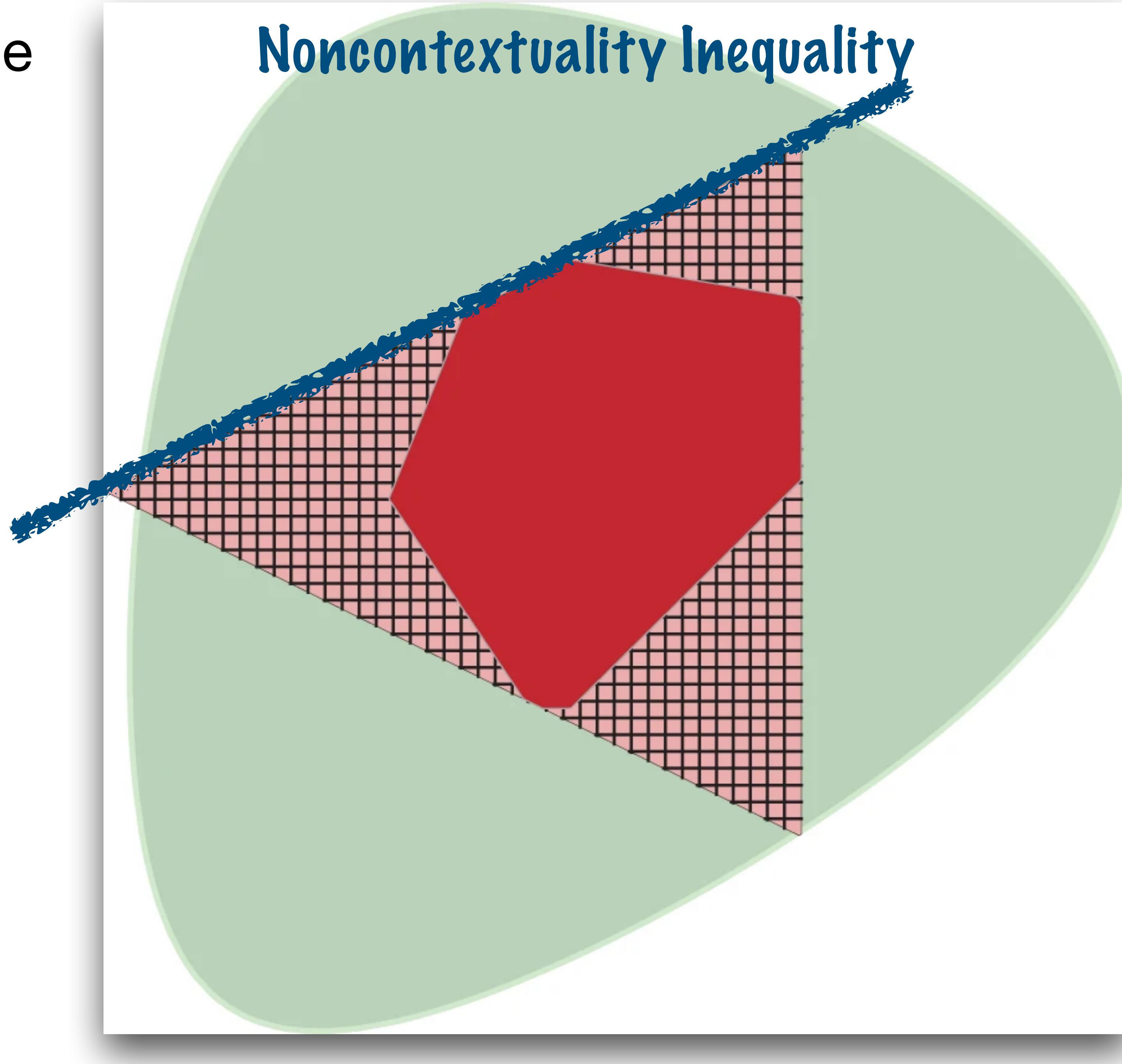

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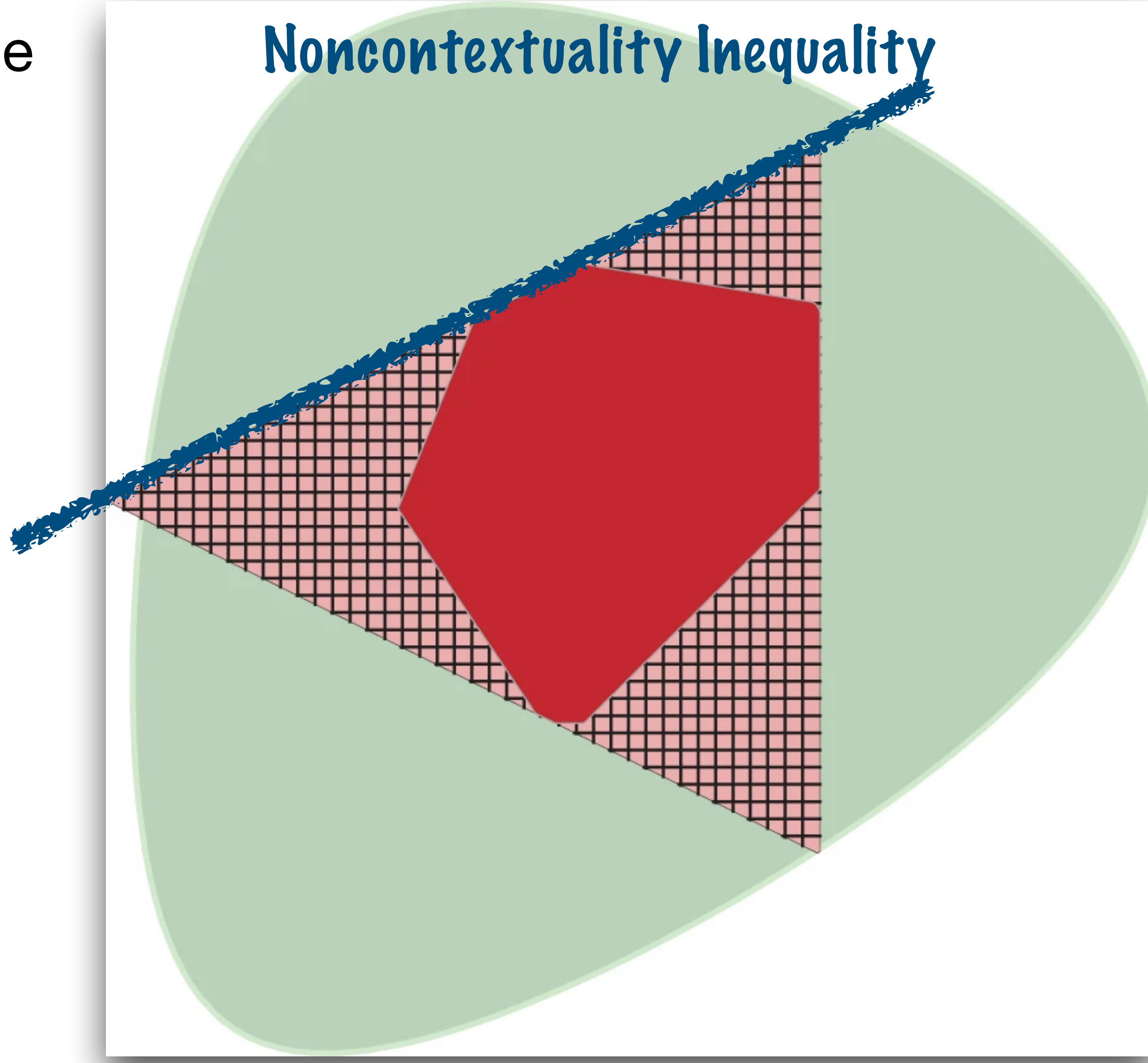

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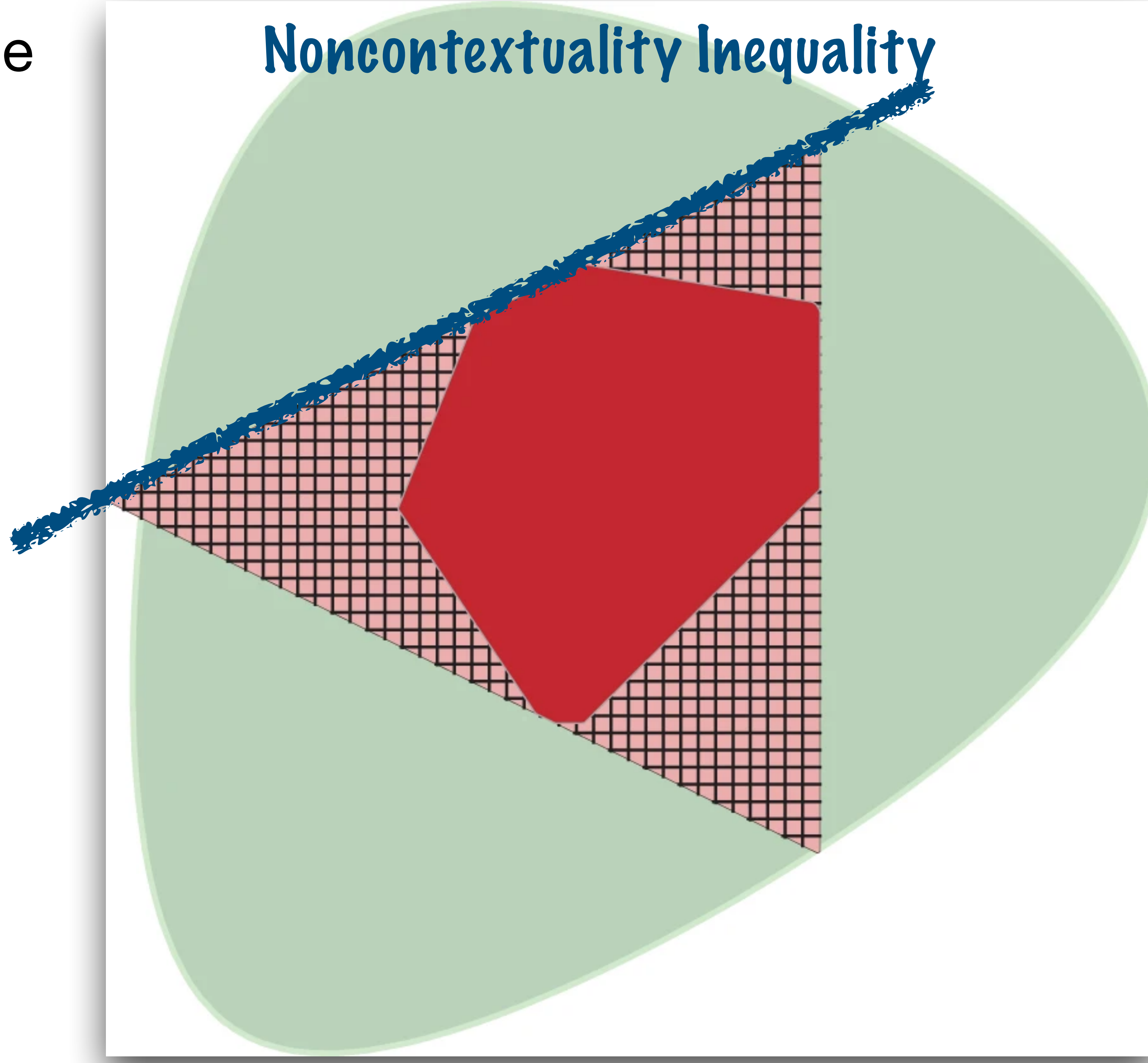

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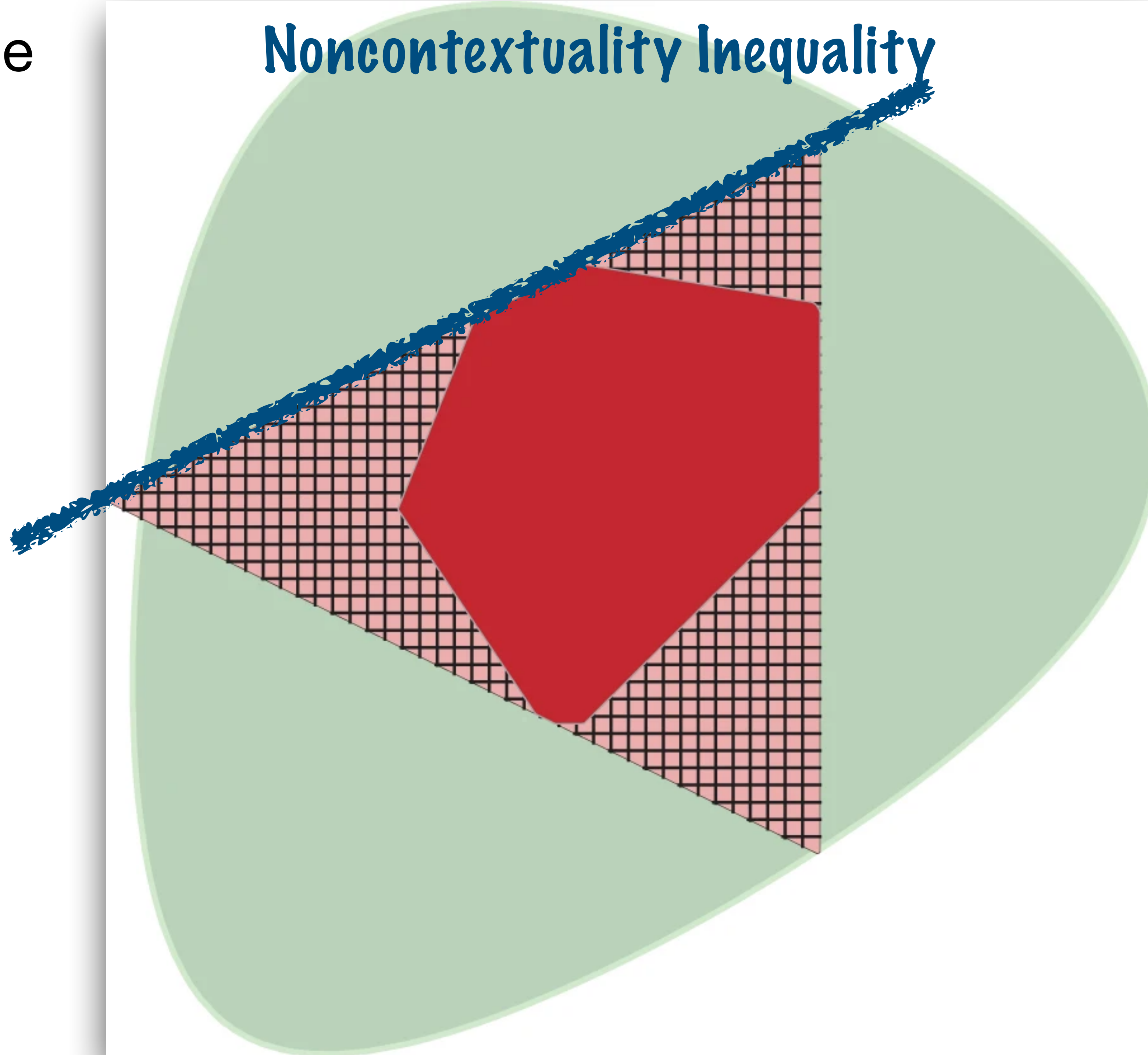
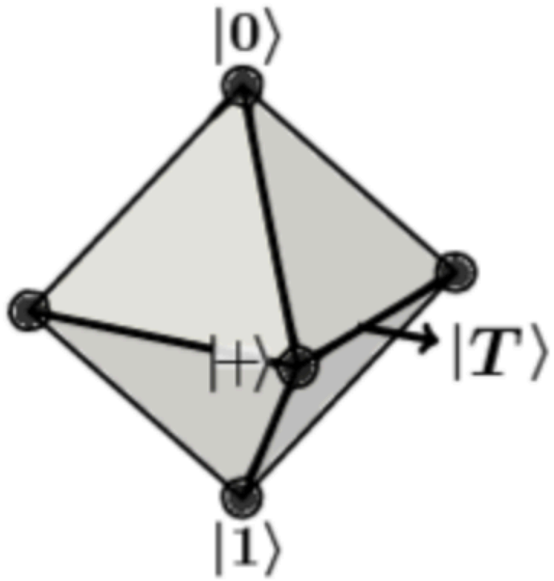


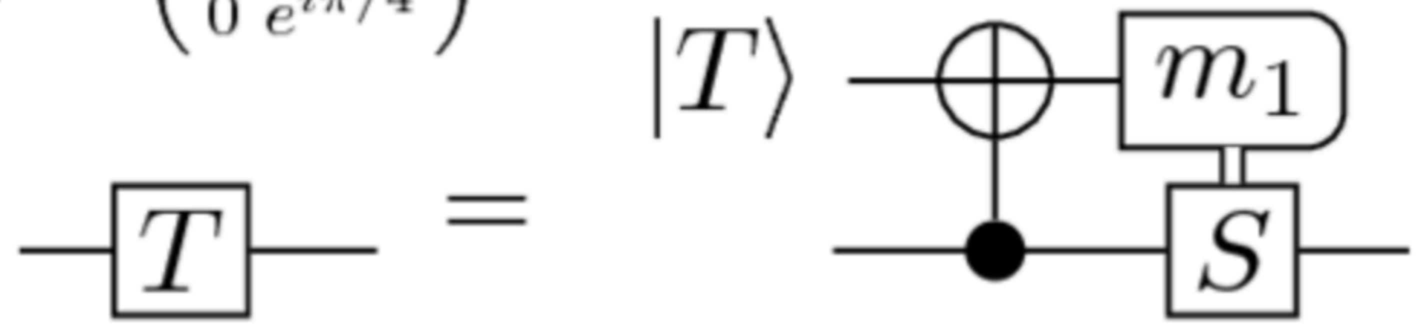

- Analagous results to qubits regarding (non-)tight distillation in Face/Edge dirns

# Proper Magic Monotones




$|T\rangle$-type magic state

$$T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$$




$|T\rangle$ states (+Cliffords) enable $T$ gates

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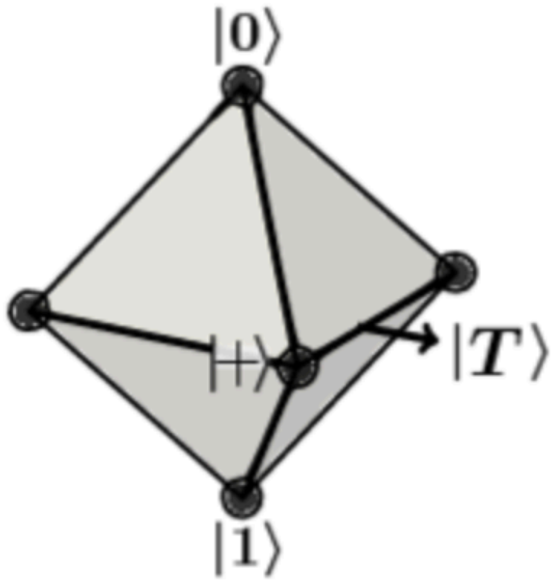


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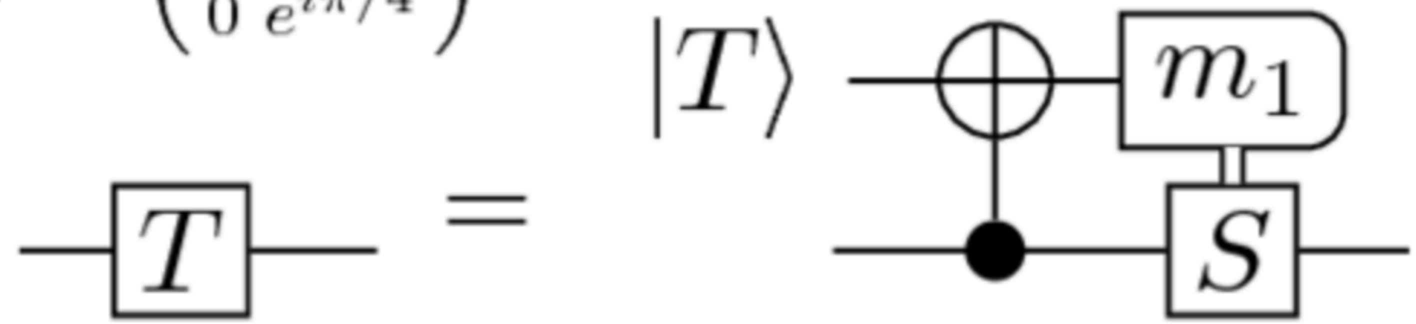


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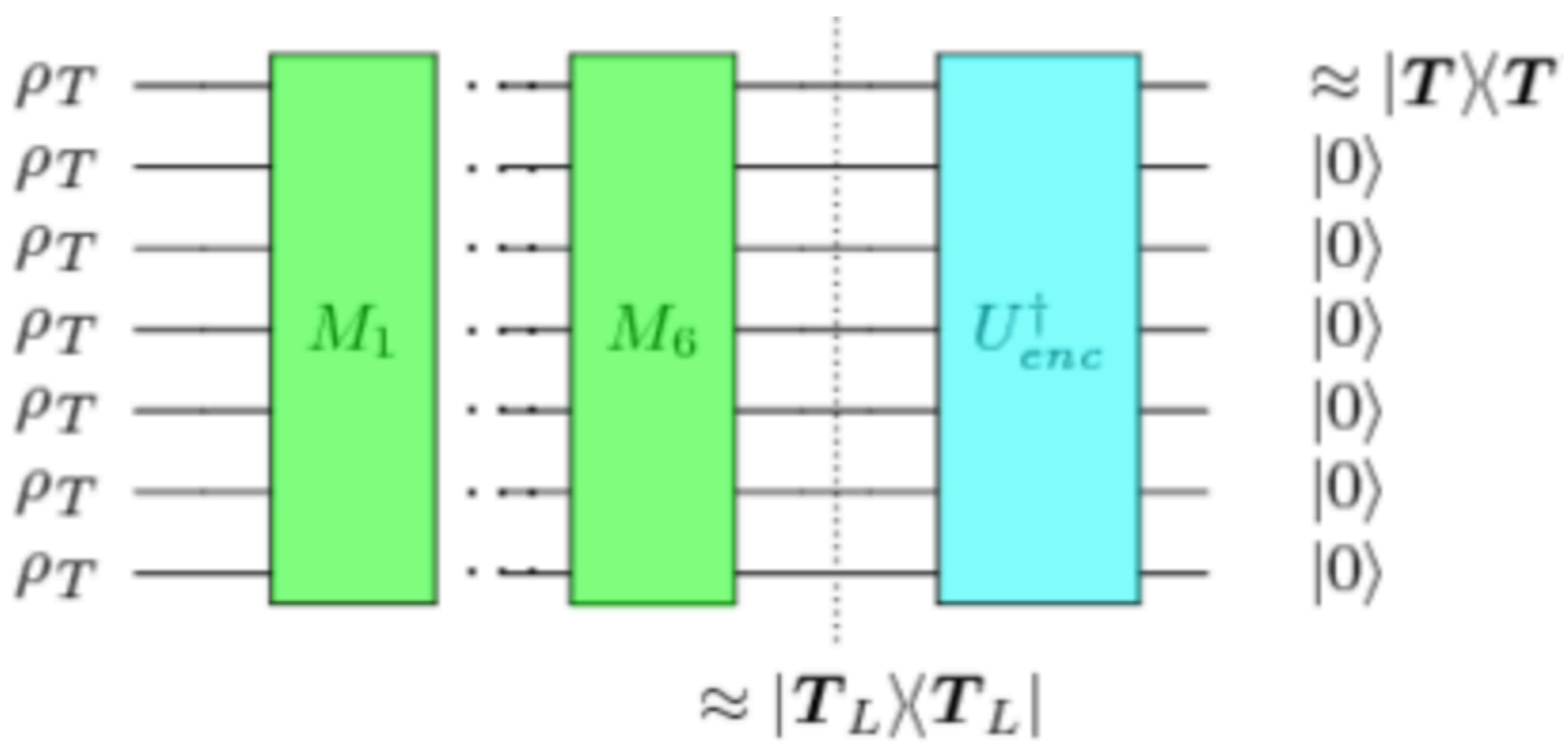

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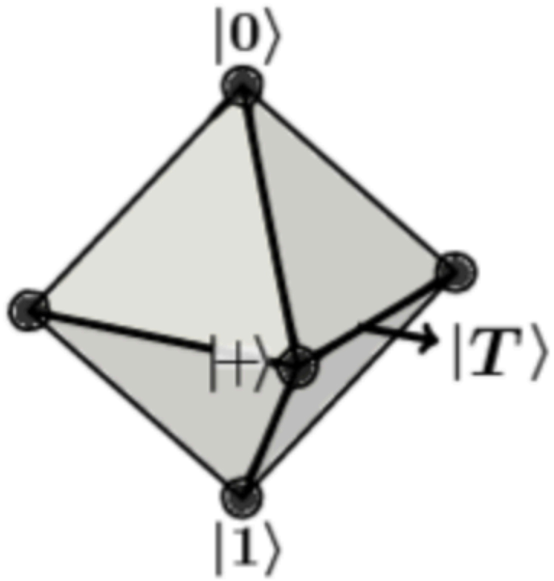


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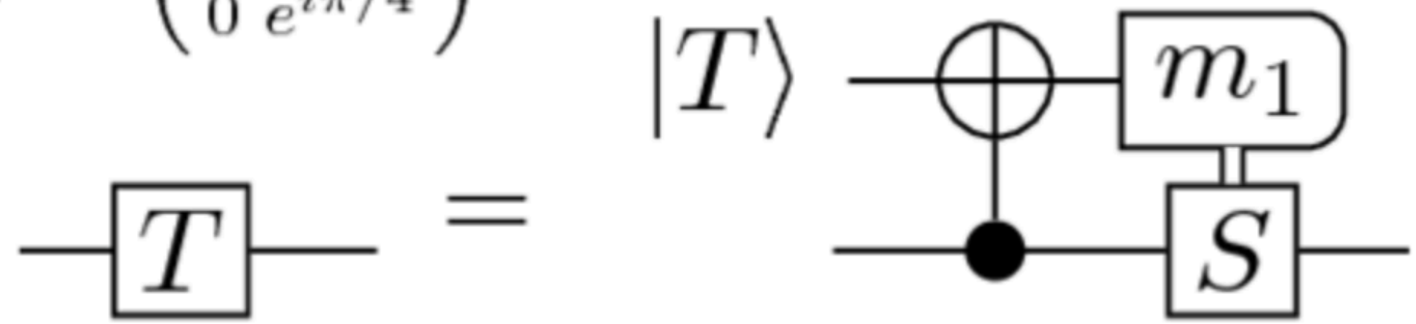


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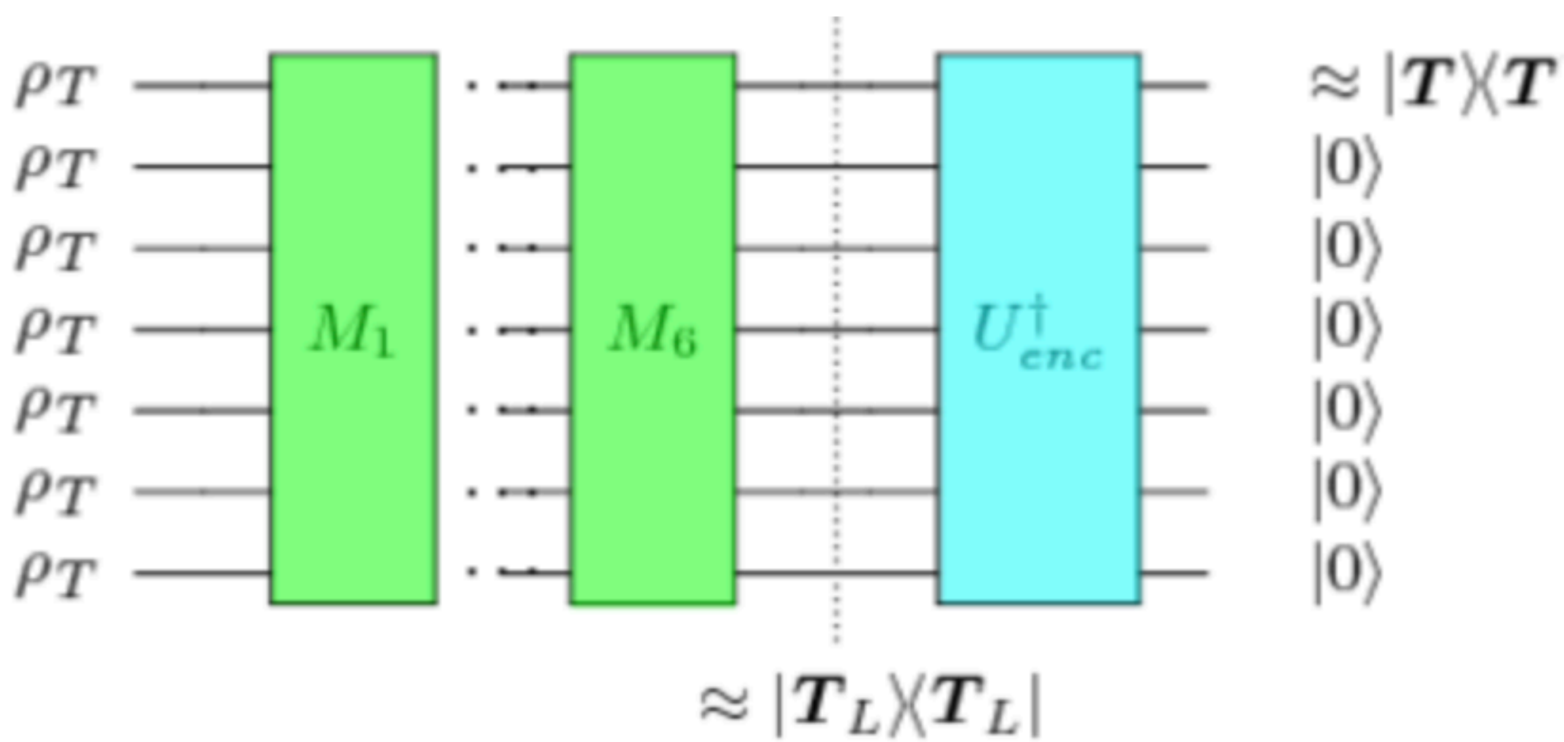
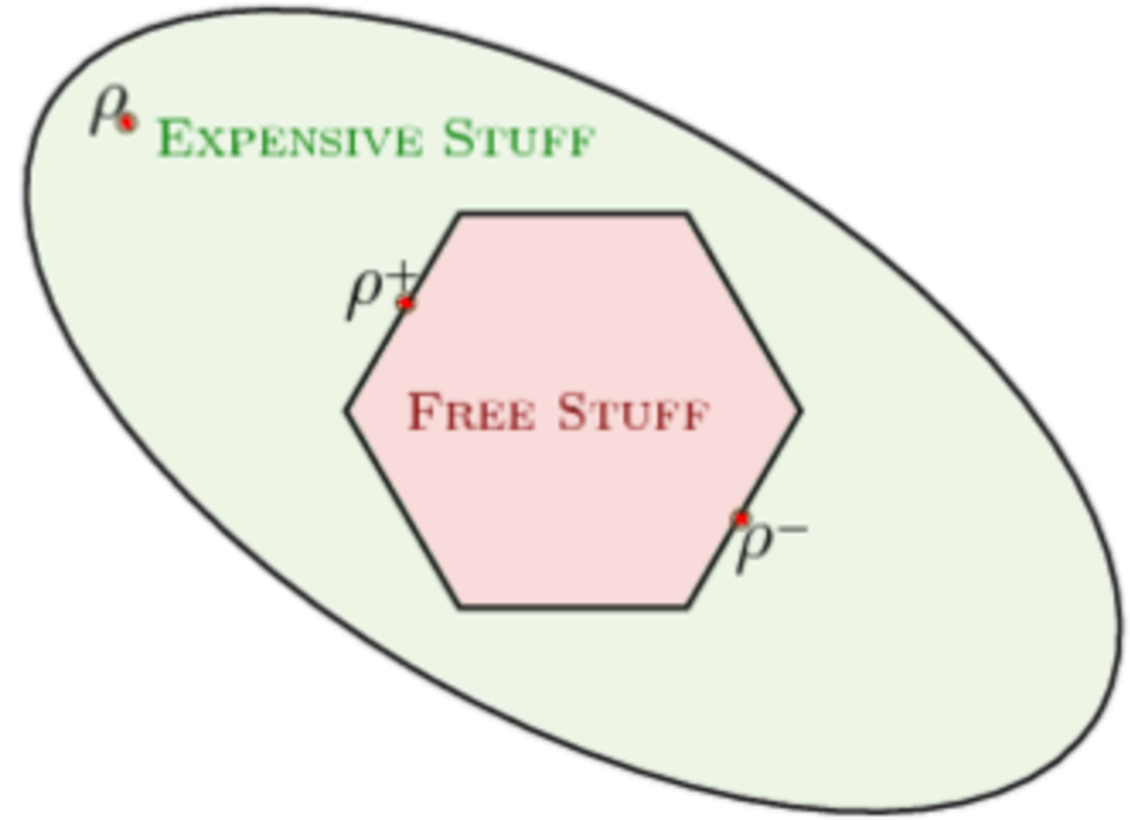
- The cost of Magic State distillation suggests a precious resource



# Robustness of Magic

- Borrowing an idea from Entanglement Theory, we can decompose a resourceful state into a linear combination of free states



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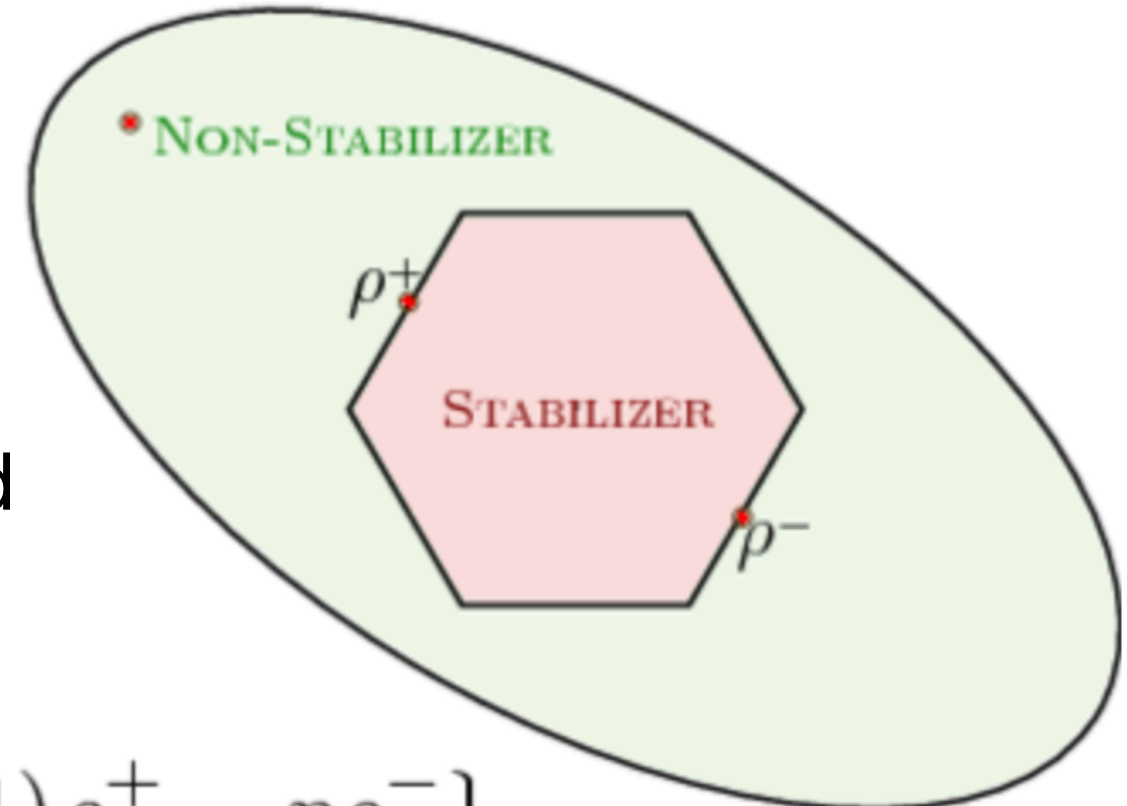


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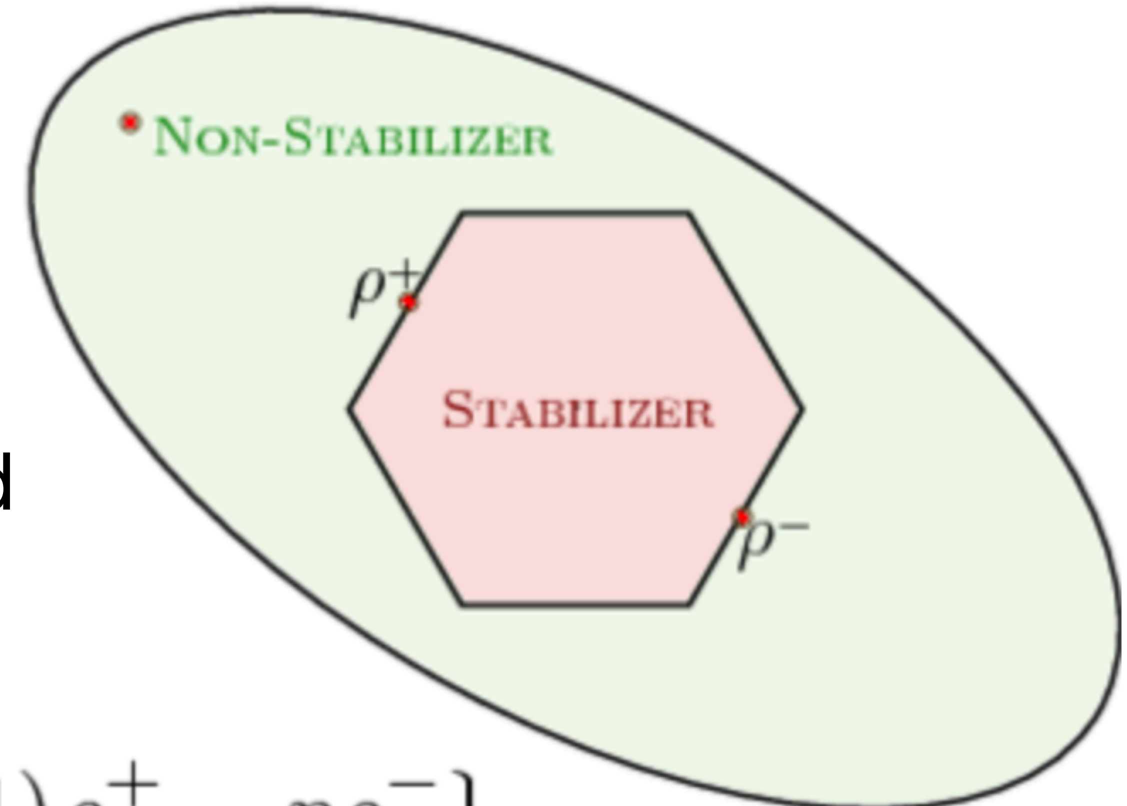


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**Resource Desiderata**

- $\mathcal{R}(\rho) \geq 1, \ (\mathcal{R}(\rho \in \mathcal{P}_{\text{STAB}}) = 1)$
- $\mathcal{R}(\rho_1 \otimes \rho_2) \leq \mathcal{R}(\rho_1)\mathcal{R}(\rho_2)$
- $\mathcal{R}\left(\mathcal{E}_{\text{STAB}}(\rho)\right) \leq \mathcal{R}\left(\rho\right)$

. . . $\Rightarrow$ Well-behaved quantifier

. . . **or take** $\log \mathcal{R}$

- $\log \mathcal{R}(\rho) \geq 0,$
- $\log \mathcal{R}(\rho_1 \otimes \rho_2) \leq \log \mathcal{R}(\rho_1) + \log \mathcal{R}(\rho_2)$
- $\log \mathcal{R}\left(\mathcal{E}_{\text{STAB}}(\rho)\right) \leq \log \mathcal{R}\left(\rho\right)$

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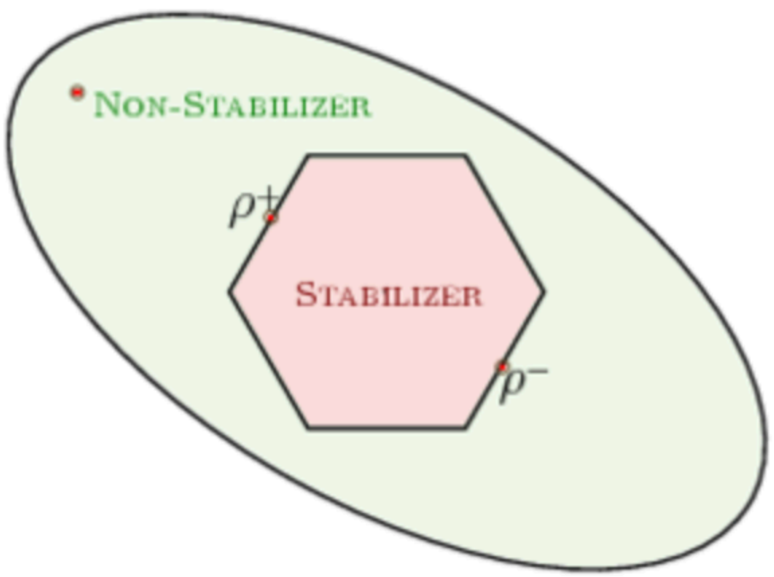
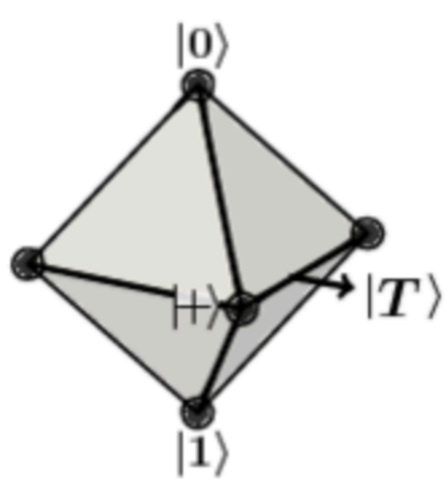
To solve the geometrical problem for $\mathcal{R}(\rho)$ rewrite as Linear Program:




$$\mathcal{R}(\rho) = \min ||x||_1 \text{ subject to } Ax = b$$

where columns of $A$ are vertices of $\mathcal{P}_{\mathrm{STAB}}$




**Prob:** $A_{\mathrm{STAB}} = \begin{matrix} \langle I \rangle \\ \langle X \rangle \\ \langle Y \rangle \\ \langle Z \rangle \end{matrix} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$

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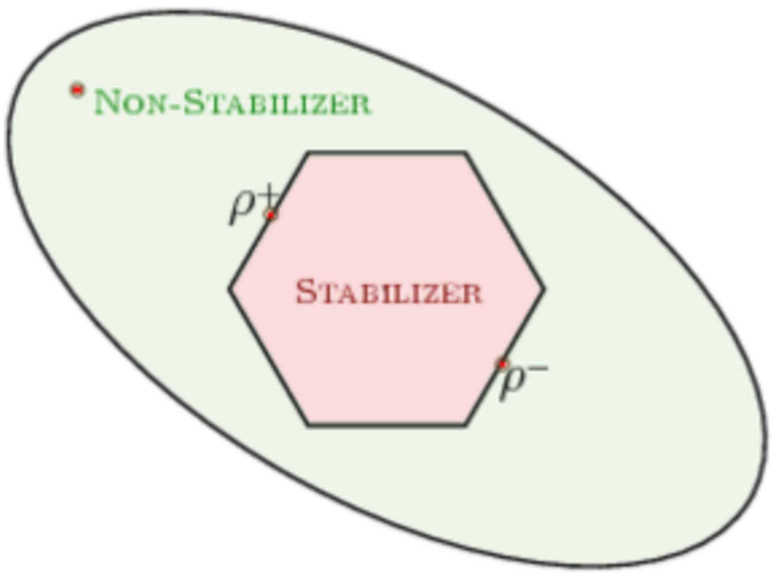


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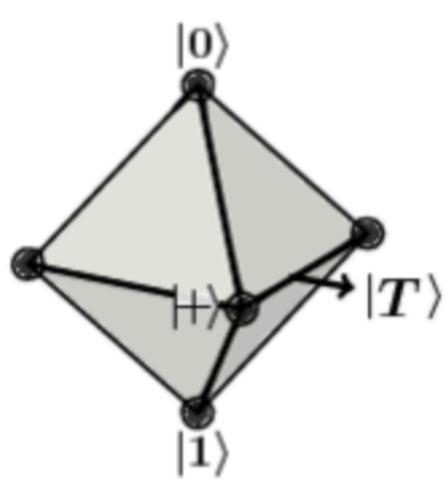


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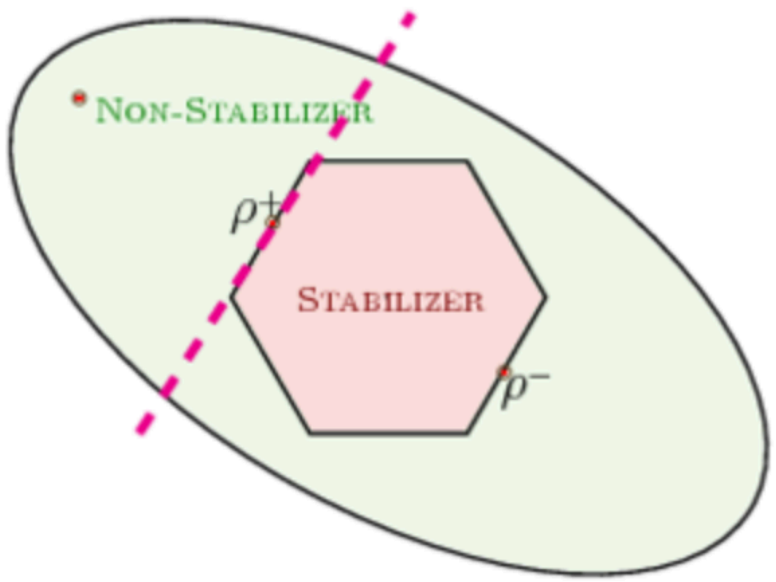


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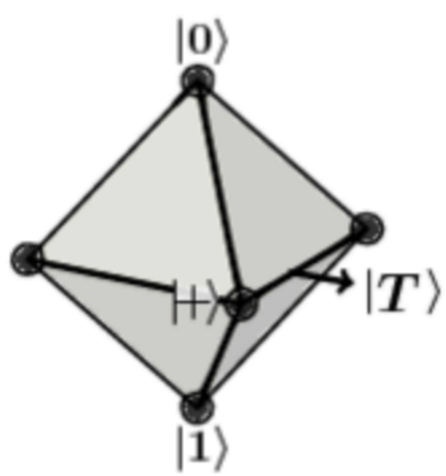


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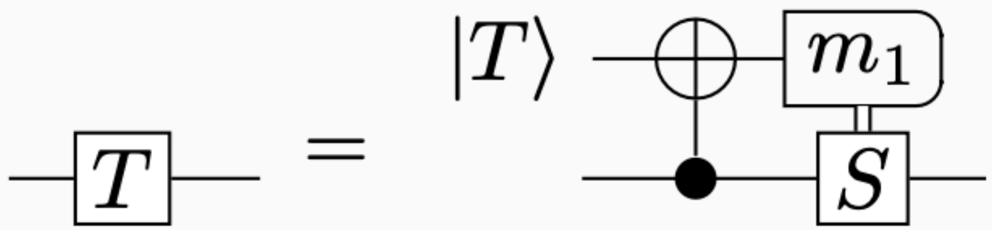
**Dual:** $\min_{Ax=b} ||x||_1 = \max_{||A^T y||_\infty \leq 1} -b^T y$ gives Witness

- Straightforward using e.g. CVX or similar
- Problem size grows rapidly in qubits: {6,60,1080,36720,2423520,…}

# Robustness of Magic

1. Realize that

   a quantum circuit with $\tau$ $T$ gates is equivalent to
   a purely Clifford circuit acting on $\tau$ magic states $|T\rangle$

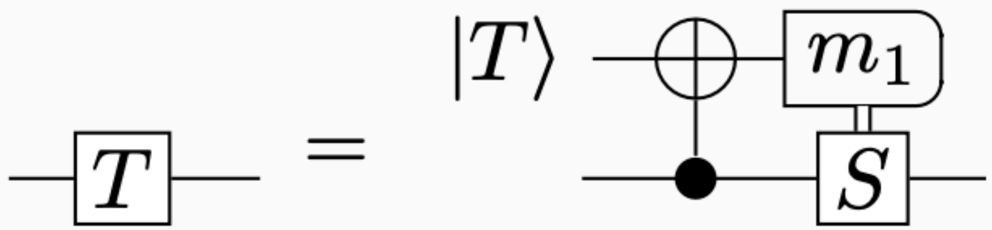



2. Adapt the efficient classical simulation schemes for Clifford circuits to allow magic state inputs e.g., input$= |0\rangle^{\otimes n-\tau}|T\rangle^{\otimes \tau}$

# Robustness of Magic

1. Realize that

   a quantum circuit with $\tau$ $T$ gates is equivalent to
   a purely Clifford circuit acting on $\tau$ magic states $|T\rangle$




2. Adapt the efficient classical simulation schemes for Clifford circuits to allow magic state inputs e.g., input$= |0\rangle^{\otimes n-\tau}|T\rangle^{\otimes\tau}$

Robustness gives a quasiprobability distribution over stabilizer states:

$$\mathcal{R}(\rho) = \min_x \left\{ \sum_i |x_i|; \rho = \sum_i x_i \, (\text{Stabilizer State})_i \right\} \quad \sum_i x_i = 1$$

Simulation takes longer to converge to desired accuracy (Chernoff-Hoeffding)

Require $\frac{2}{\delta^2}(\sum_i |x_i|)^2 \ln\left(\frac{2}{\epsilon}\right)$ samples to get $\delta$-close to real dist. with prob $1-\epsilon$

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$$\mathcal{R}(|T\rangle)^{2\tau} \sim 2^{\tau}$$

$$\mathcal{R}(|T,T\rangle)^{\frac{2\tau}{2}} \sim 1.748^{\tau}$$

$$\mathcal{R}(|T,T,T\rangle)^{\frac{2\tau}{3}} \sim 1.701^{\tau}$$

$$\mathcal{R}(|T,T,T,T\rangle)^{\frac{2\tau}{4}} \sim 1.692^{\tau}$$

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- Morally similar to $Ax = b$ calculation for Robustness, except rows of $A$ are now stabilizer kets. Optimisation is a SOCP.

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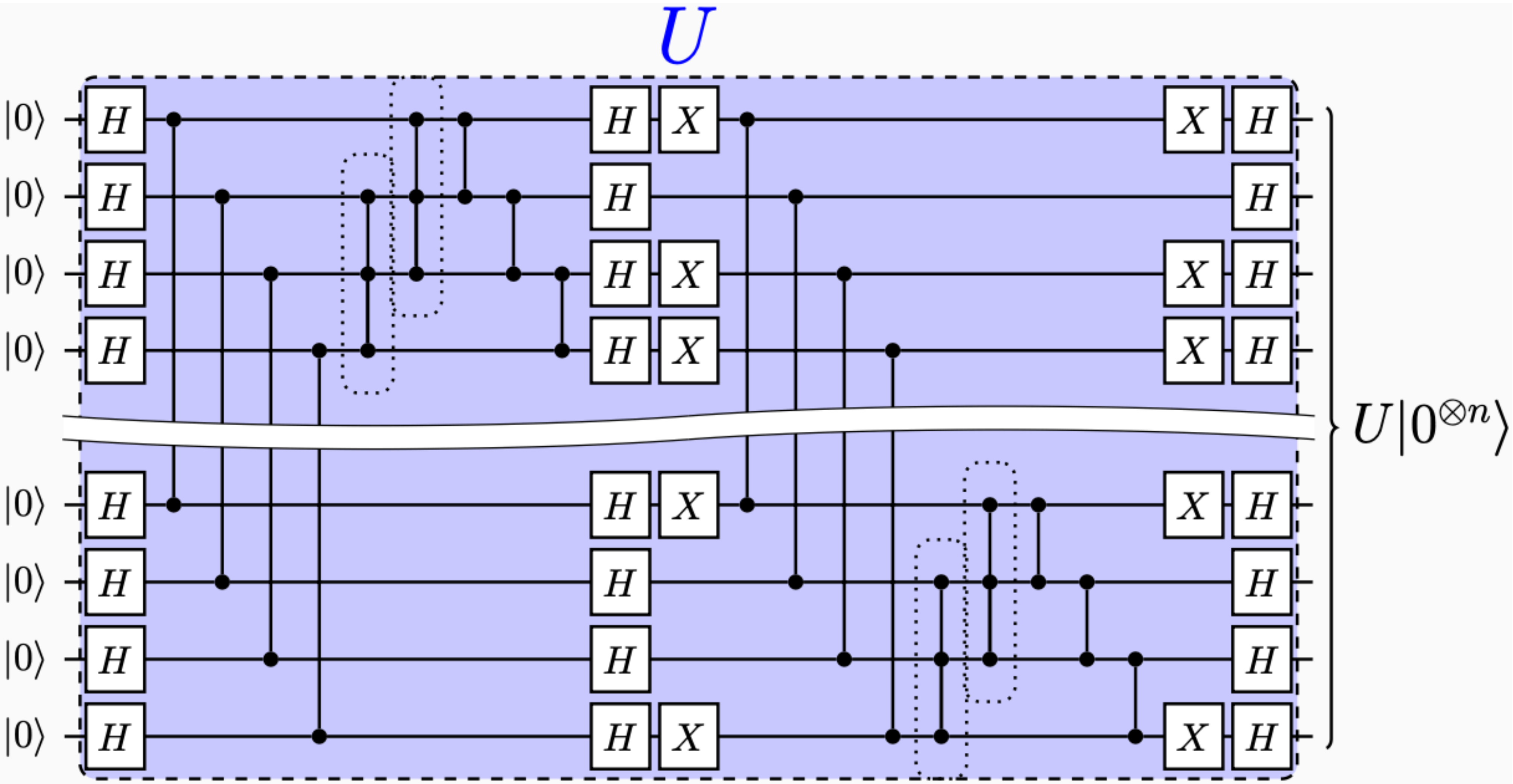

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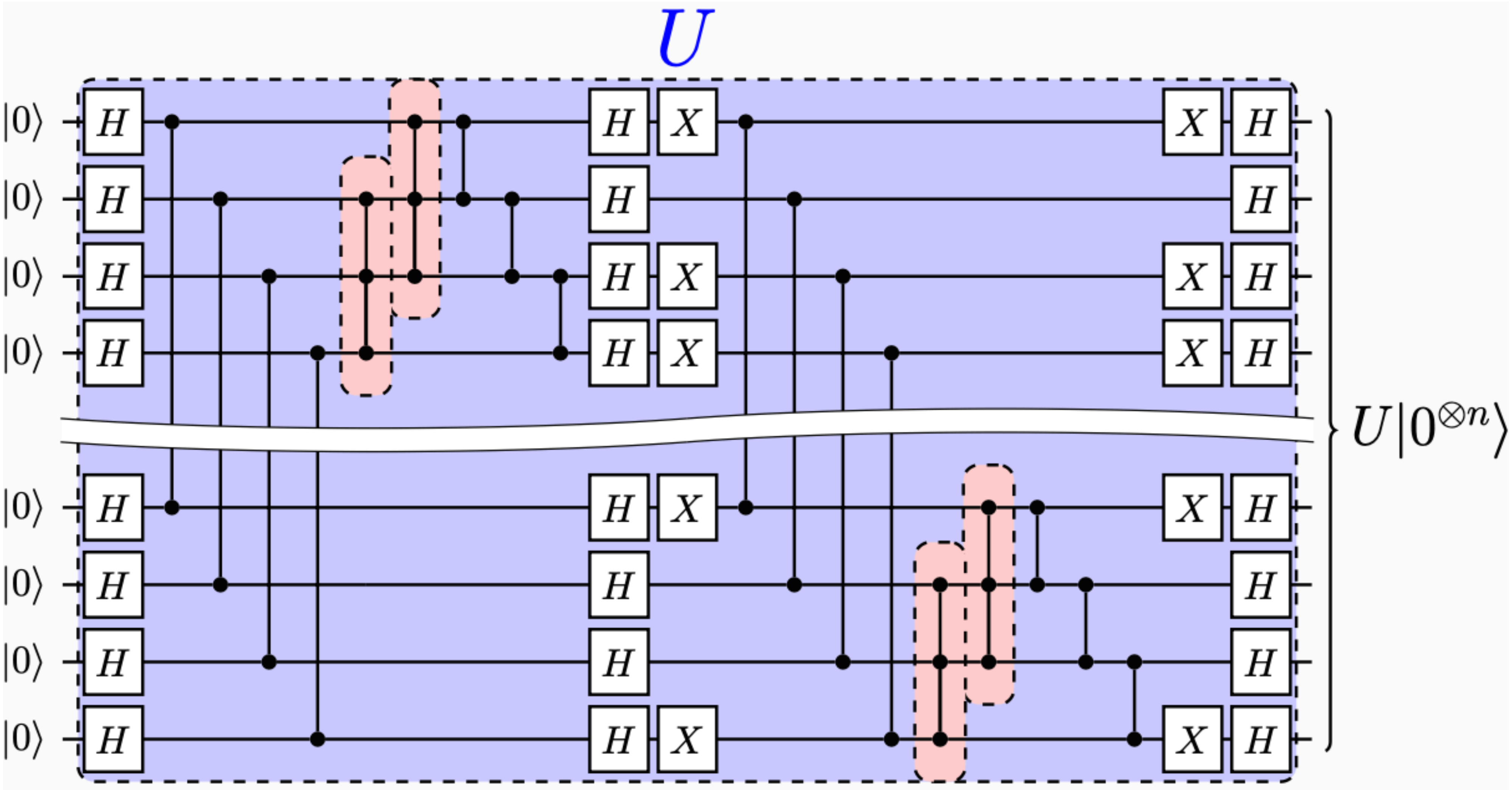

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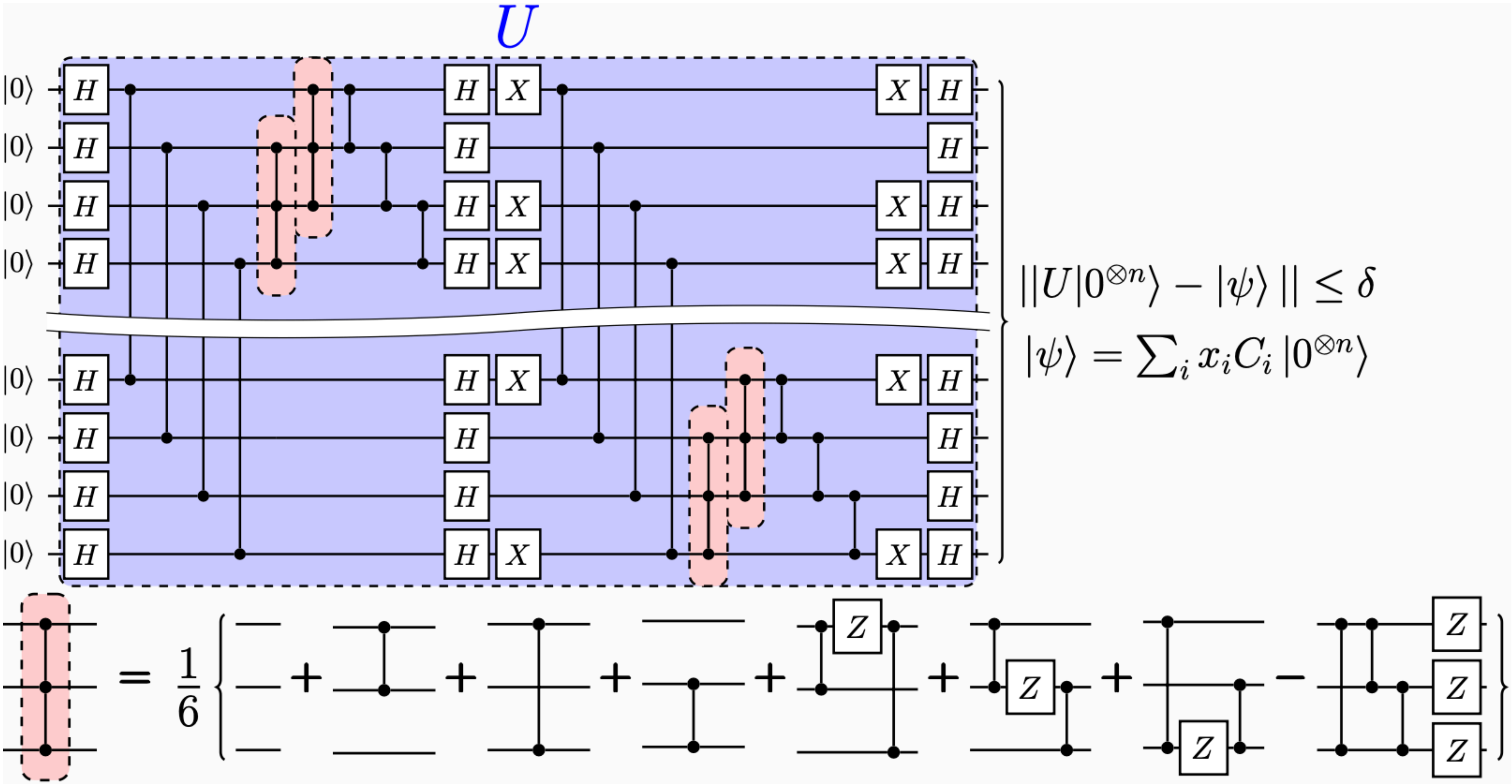

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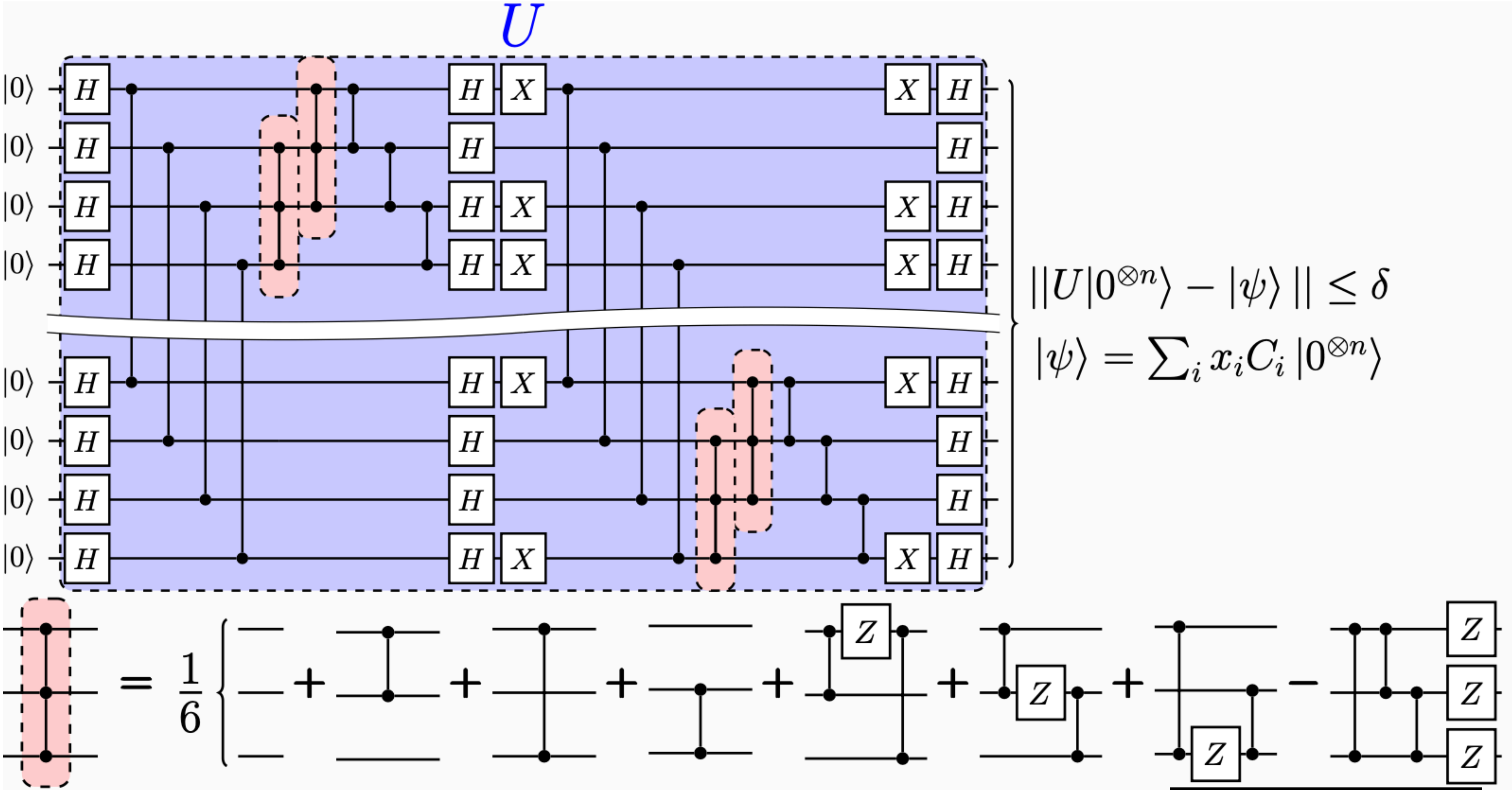


- Can decompose diagonal non-Cliffords into Cliffords
- E.g. Every time we encounter a CCZ, roll a D8
- Close to true w.h.p if choose $(||x||_1/\delta)^2$ paths

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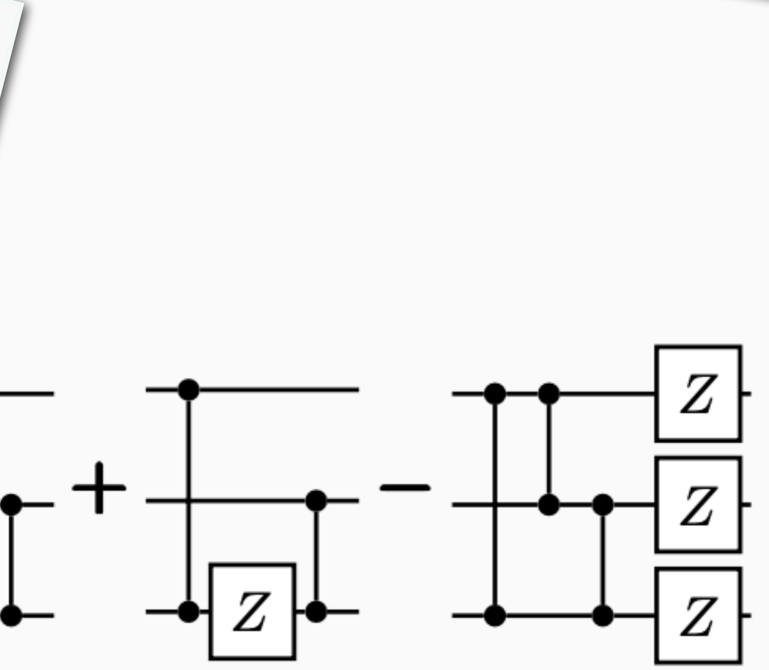
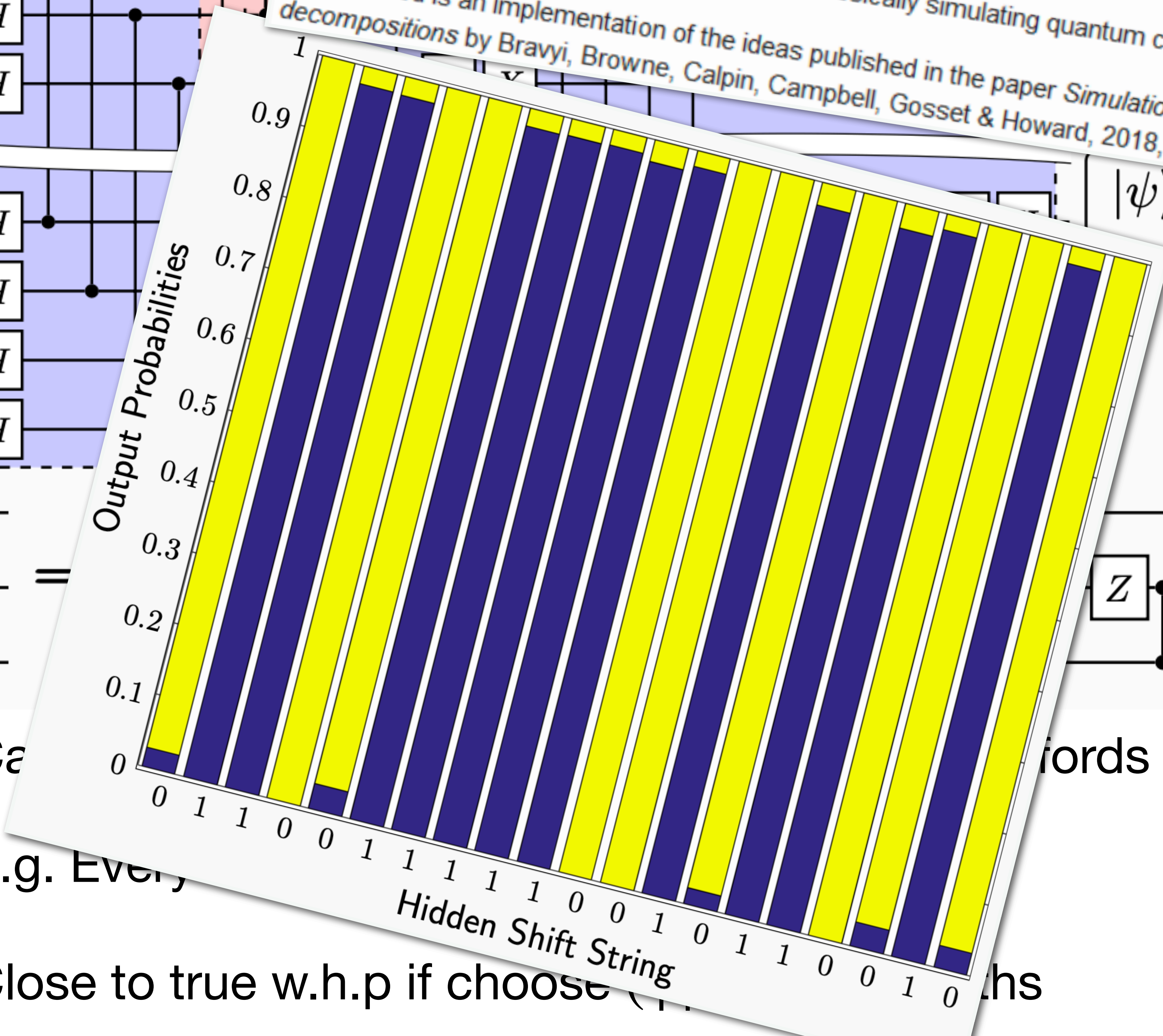



# The Extended Stabilizer Simulator

## Introduction

The Extended Simulator is a new method for classically simulating quantum circuits available in the latest release of Qiskit-Aer.

This method is an implementation of the ideas published in the paper *Simulation of quantum circuits by low-rank stabilizer decompositions* by Bravyi, Browne, Calpin, Campbell, Gosset & Howard, 2018, arXiv:1808.00128.

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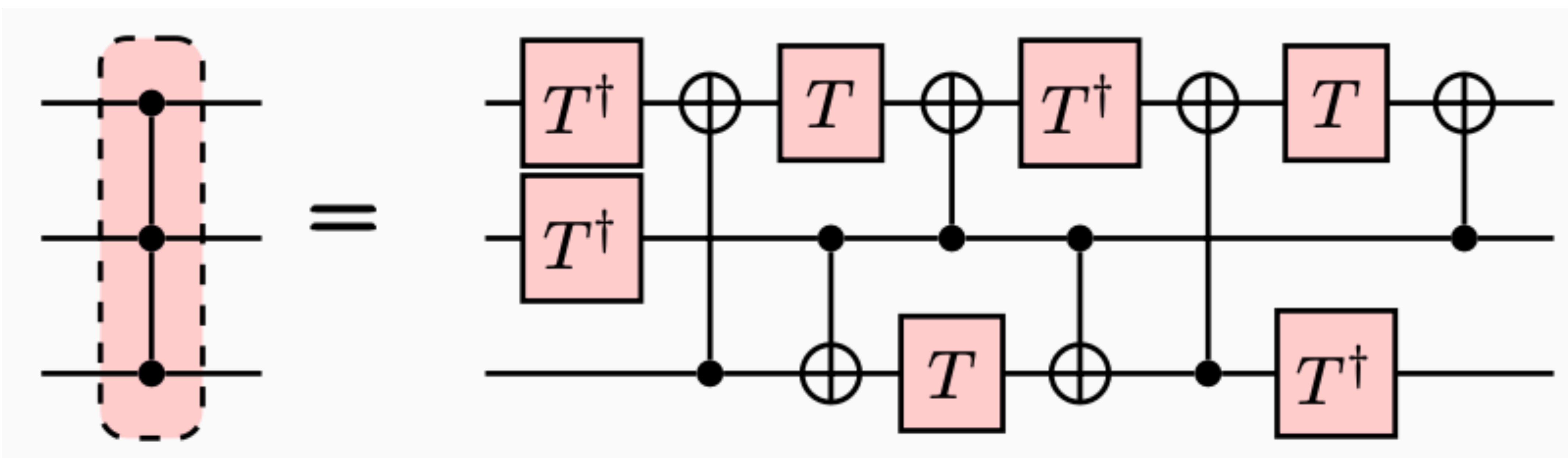

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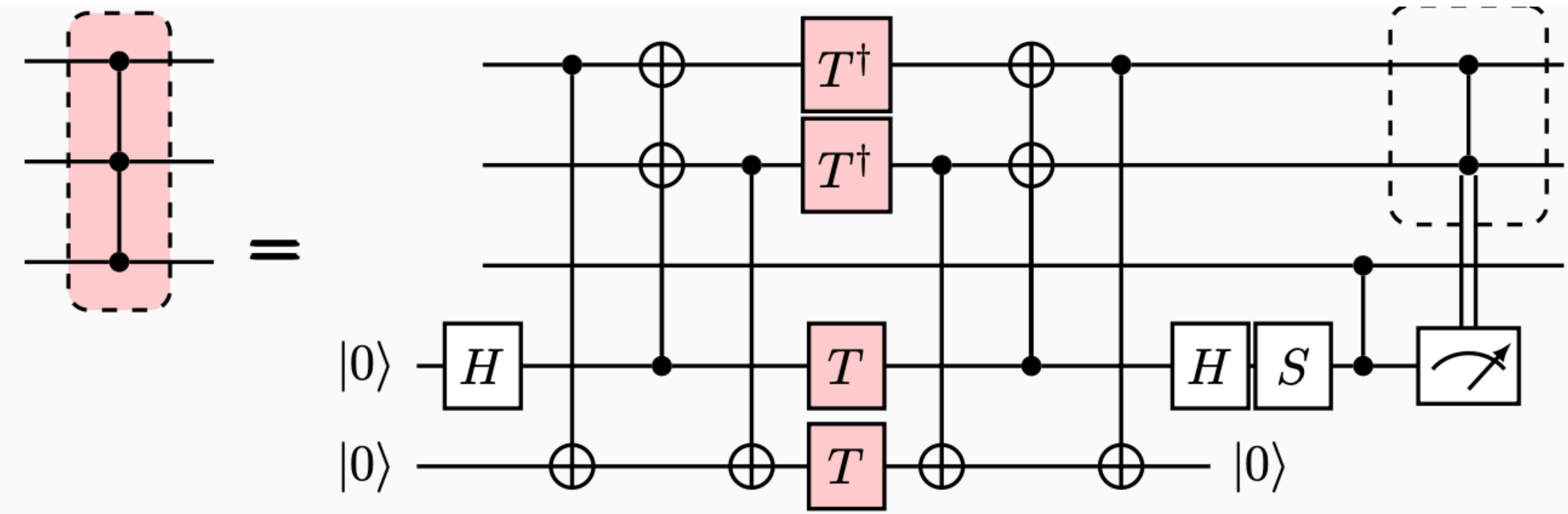


- Ancilla-assisted synthesis is powerful, practical but hard to prove

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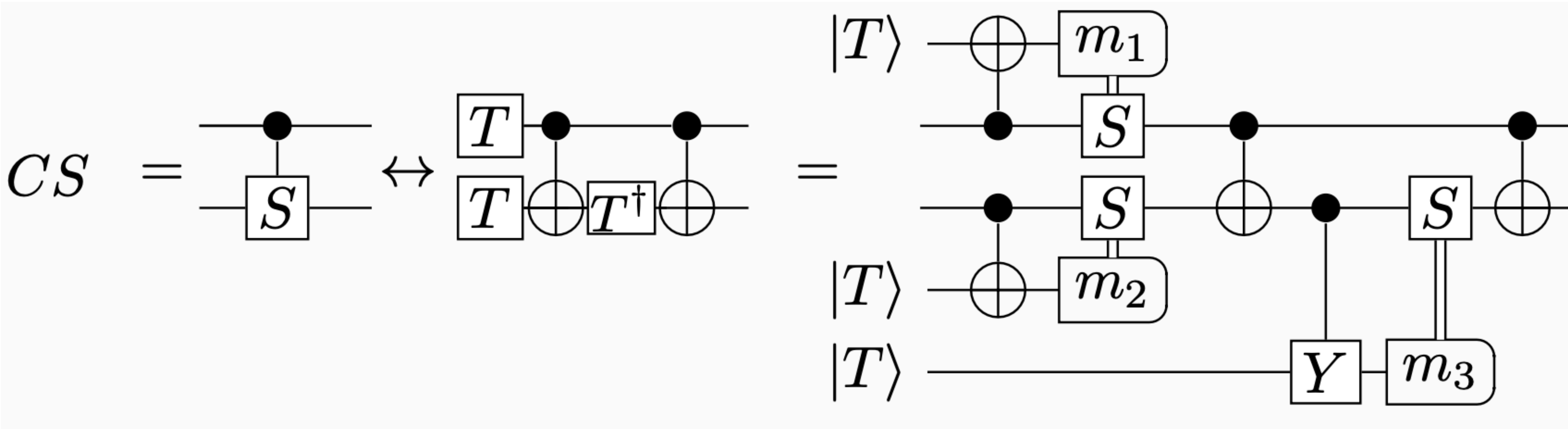

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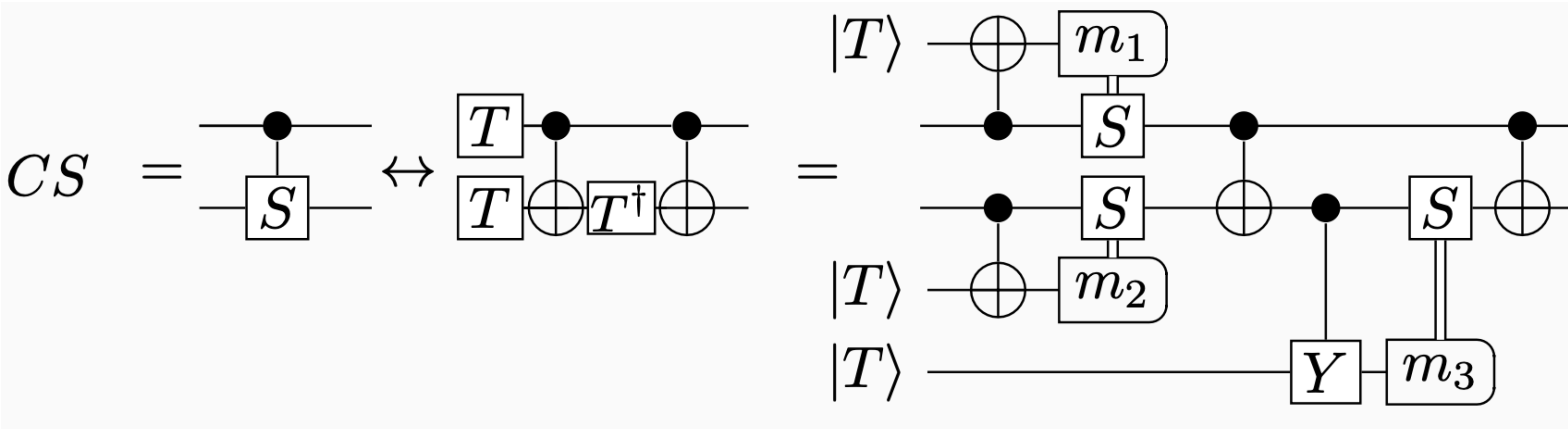


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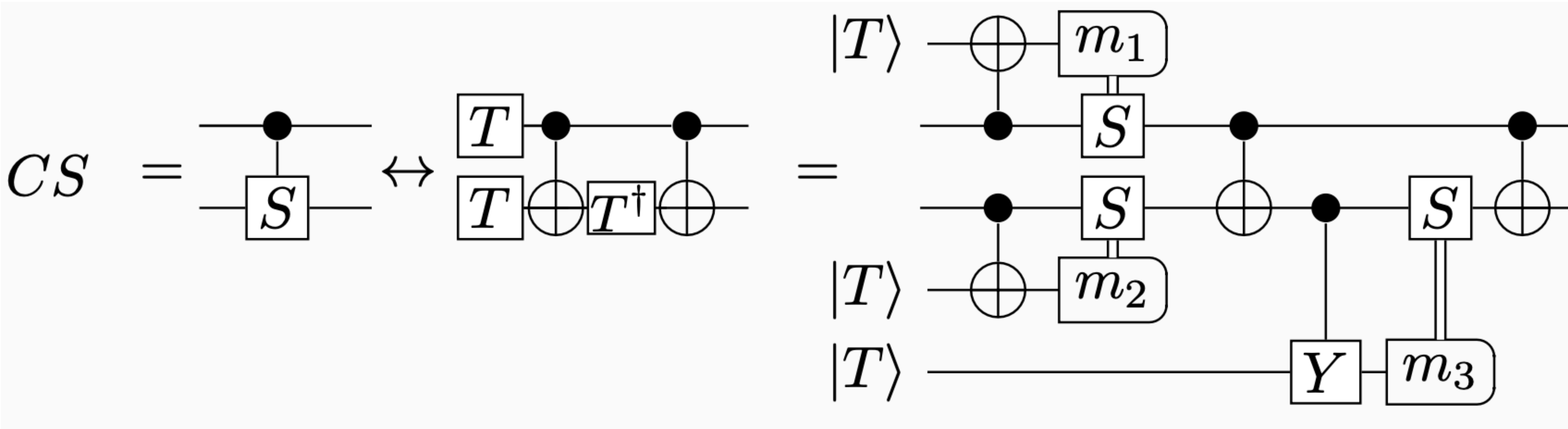


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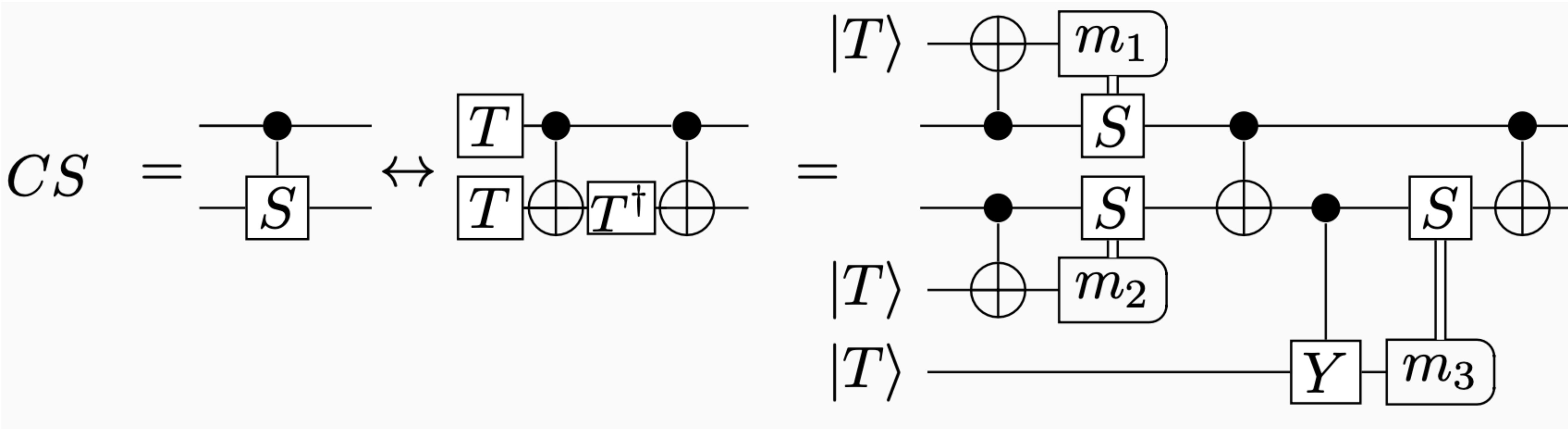


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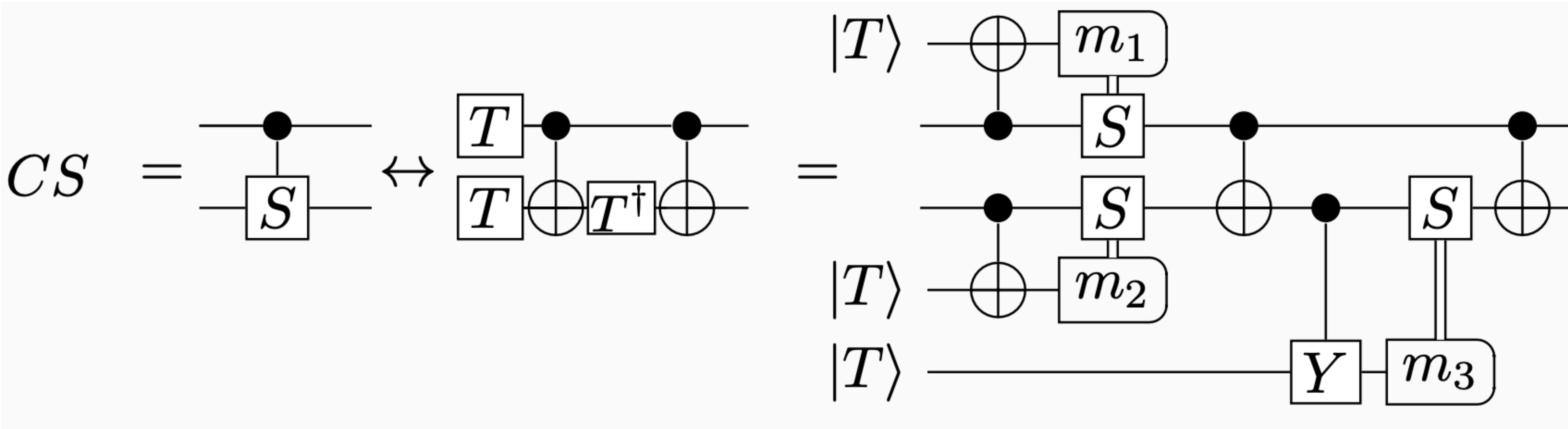

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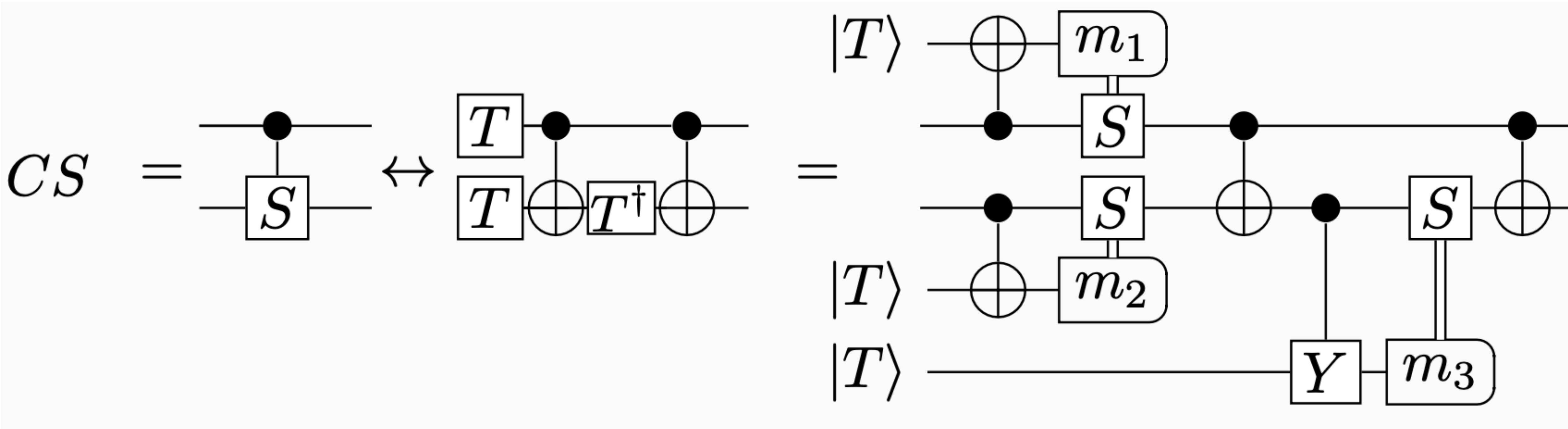


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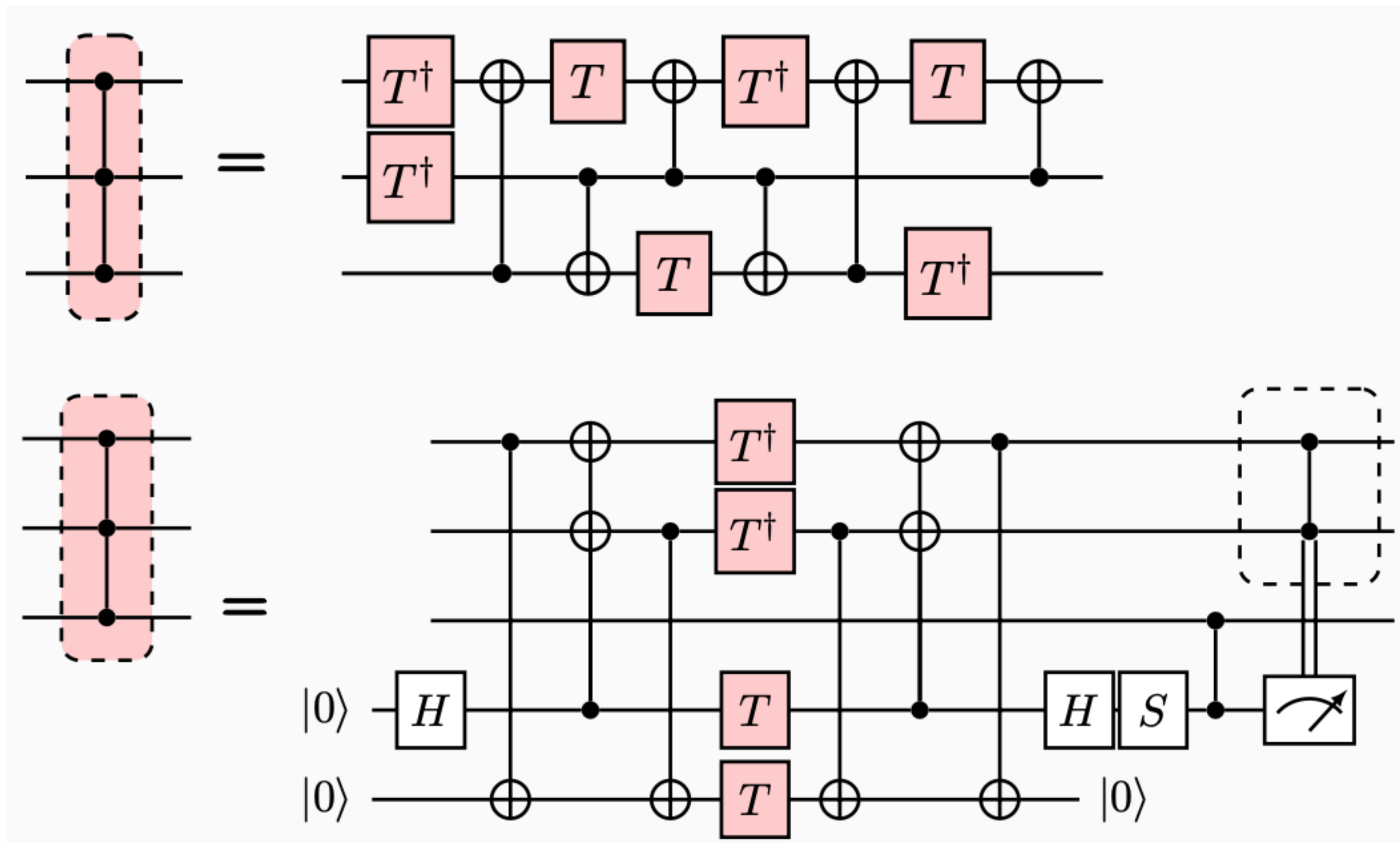


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- **<u>Meaning:</u>** impossible to compile $CCZ$ with fewer than 4 $T$ gates

- The above construction is $T$-optimal

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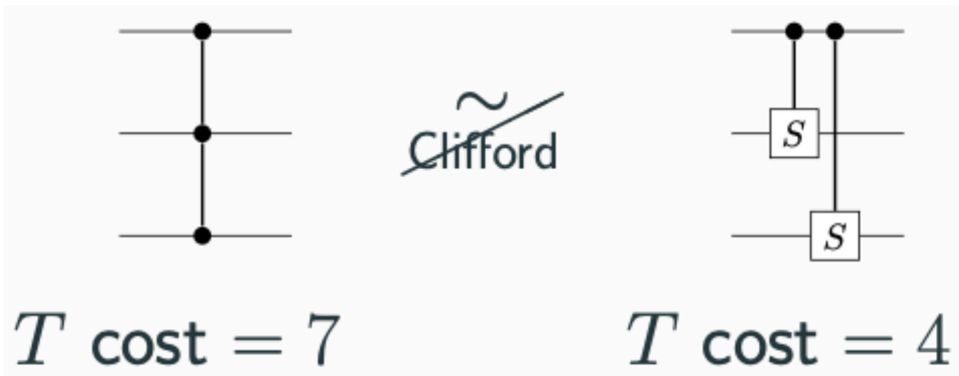


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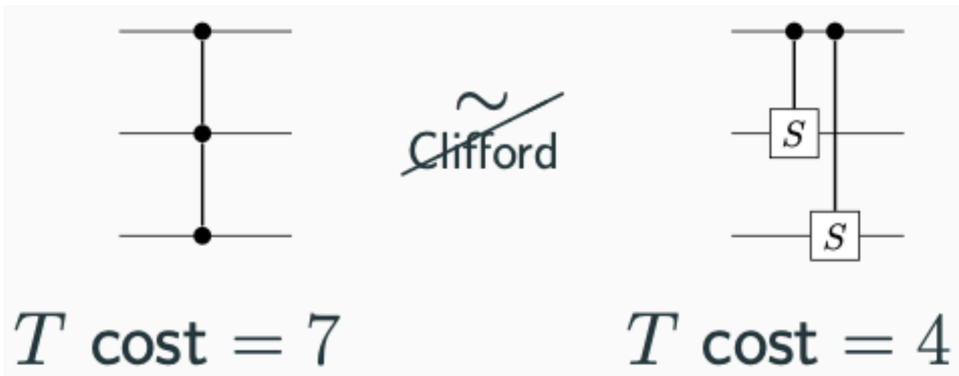



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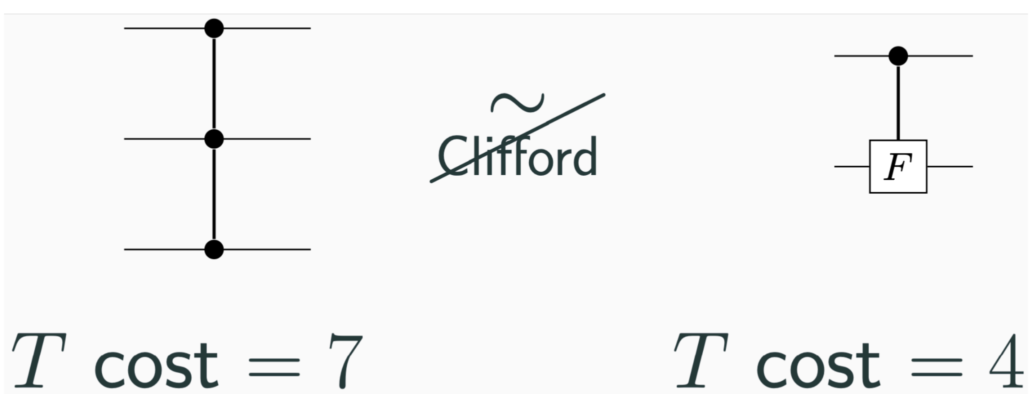


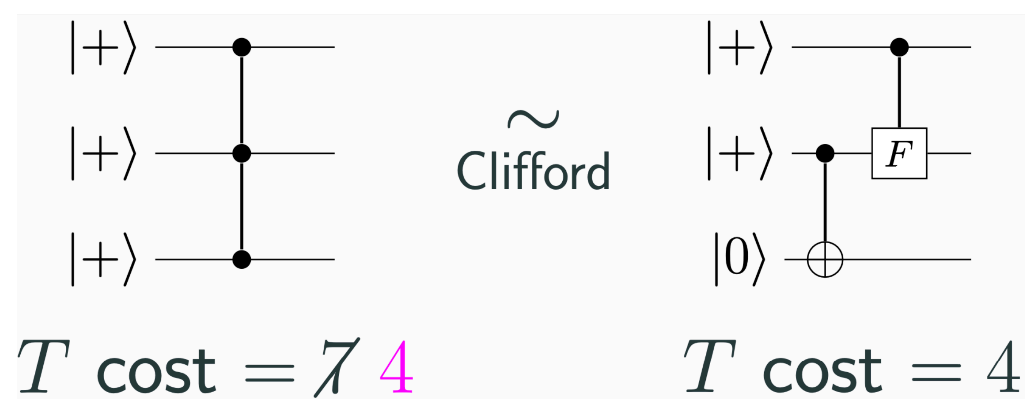

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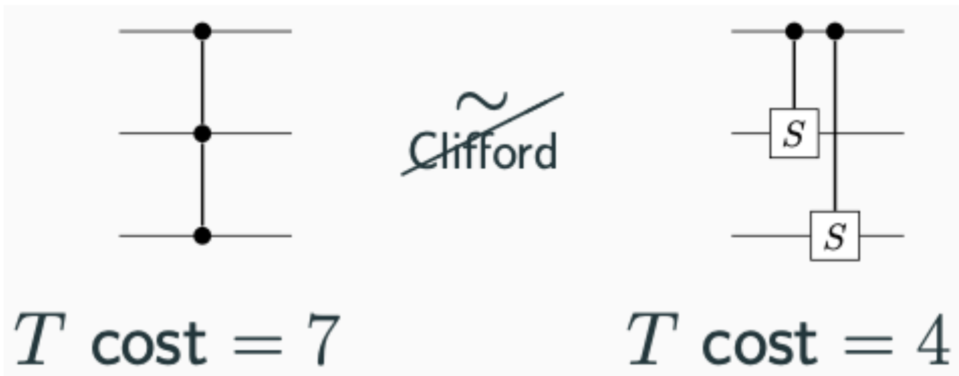



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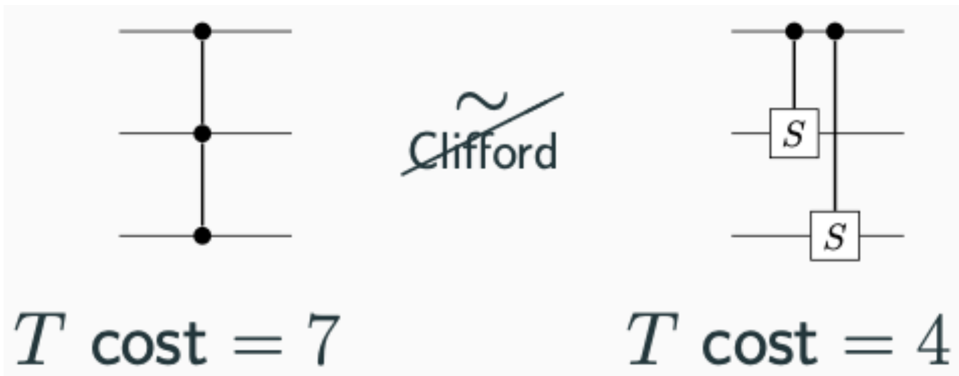



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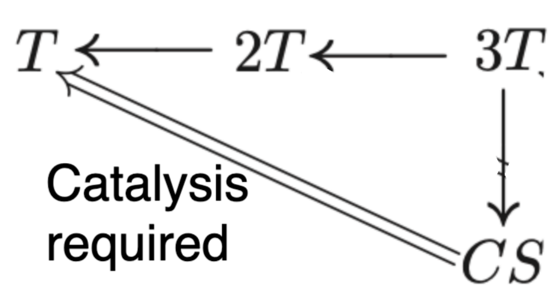

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# Slide Title

- First